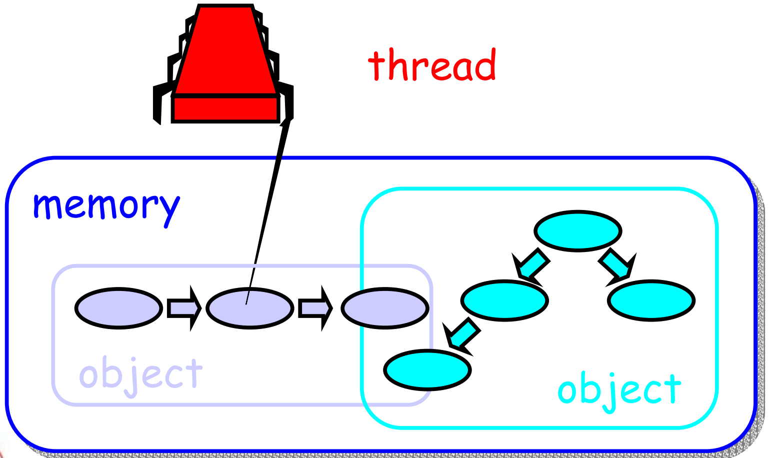
The Consensus Problem

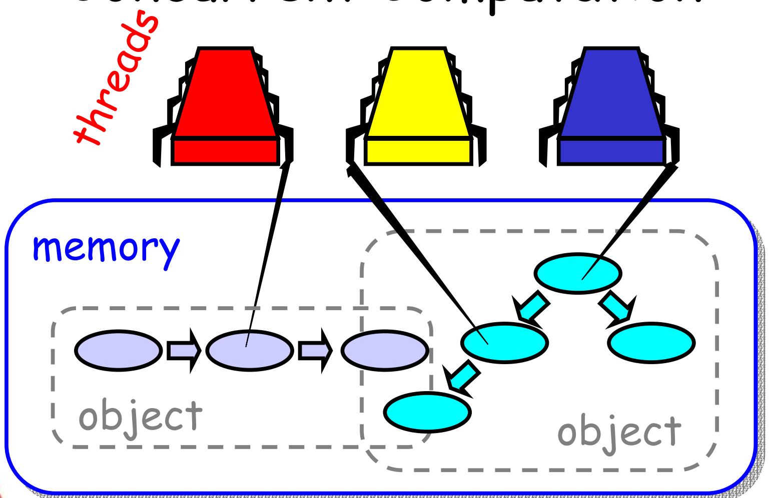
Roger Wattenhofer

a lot of kudos to
Maurice Herlihy
and Costas Busch
for some of
their slides

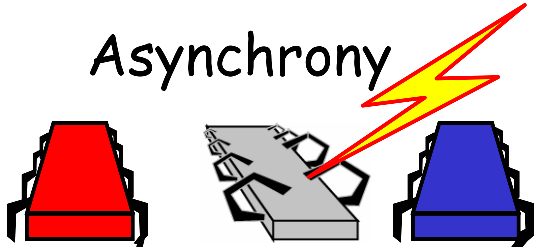
Sequential Computation



Concurrent Computation



Asynchrony



Sudden unpredictable delays
- Cache misses (*short*)
- Page faults (*long*)
- Scheduling quantum used up (*really long*)

Model Summary

- Multiple *threads*
  - Sometimes called *processes*
- Single shared *memory*
- *Objects* live in memory
- Unpredictable asynchronous delays

Road Map

- We are going to focus on principles
  - Start with idealized models
  - Look at a simplistic problem
  - Emphasize correctness over pragmatism
  - "Correctness may be theoretical, but incorrectness has practical impact"

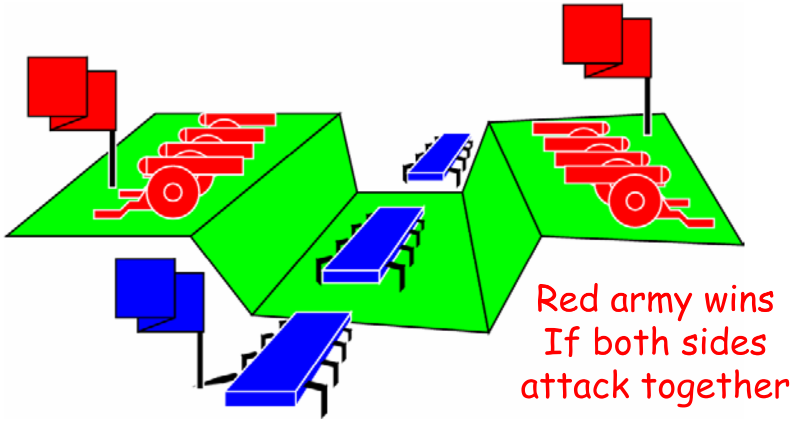
You may ask yourself …

I'm no theory weenie - why all the theorems and proofs?

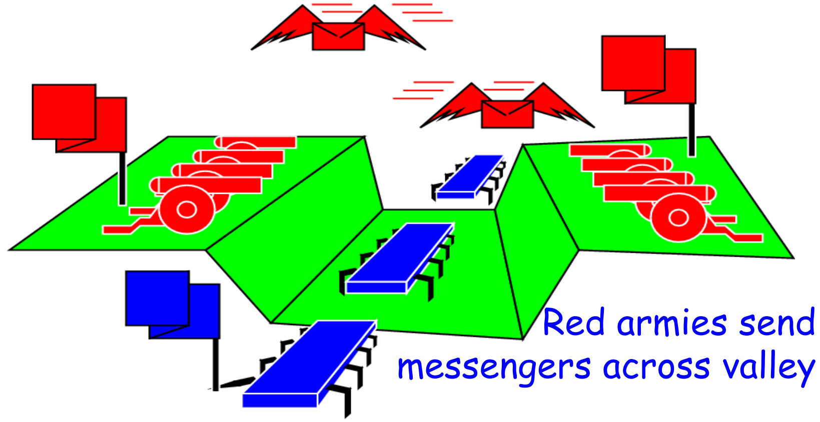
Fundamentalism

- Distributed & concurrent systems are *hard*
  - Failures
  - Concurrency
- Easier to go from theory to practice than vice-versa

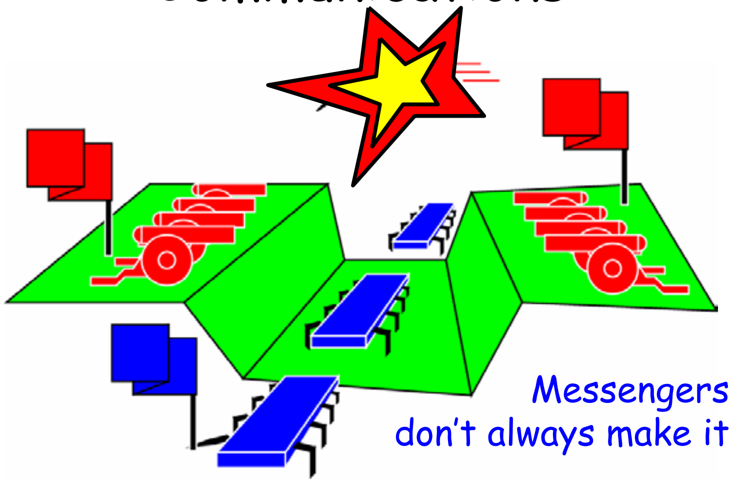
The Two Generals



Communications



Communications



Your Mission

Design a protocol to ensure that red armies attack simultaneously

Theorem

There is no non-trivial protocol that ensures the red armies attacks simultaneously

Proof Strategy

- Assume a protocol exists
- Reason about its properties
- Derive a contradiction

Proof

1. Consider the protocol that sends fewest messages
2. It still works if last message lost
3. So just don't send it
   - Messengers' union happy
4. But now we have a shorter protocol!
5. Contradicting #1

Fundamental Limitation

- Need an unbounded number of messages
- Or possible that no attack takes place

You May Find Yourself …

I want a real-time YAFA compliant Two Generals protocol using UDP datagrams running on our enterprise-level fiber tachyion network ...

You might say

I want a real-time YAFA
p
r
fiber tachyion netwo

Yes, Ma'am, right away!

You might say

Advantage:
- Buys time to find another job
- No one expects software to work anyway

fiber tachyion netwo

You might say

Advantage:
- Buys time to find another job
- No
any

Disadvantage:
- You're doomed
- Without this course, you may not even know you're doomed

You might say

I can't find a fault-tolerant algorithm, I guess I'm just a pathetic loser.

You might say

Advantage:

- No need to take course

algorithm, I guess I'm just a pathetic loser

You might say

Advantage:

Disadvantage:

- Boss fires you, hires University St. Gallen graduate

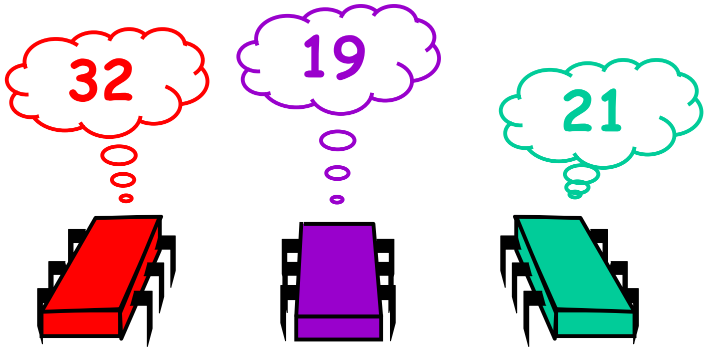
You might say

I want a real-time YAFA

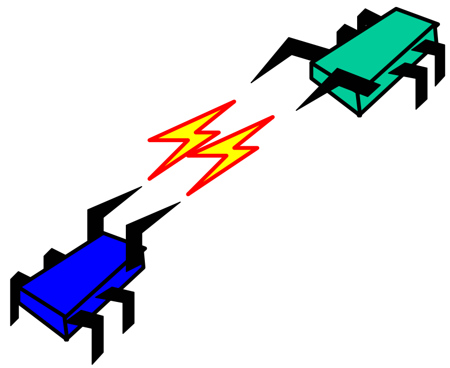
Using skills honed in course, I can avert certain disaster!

- Rethink problem spec, or
- Weaken requirements, or
- Build on different platform

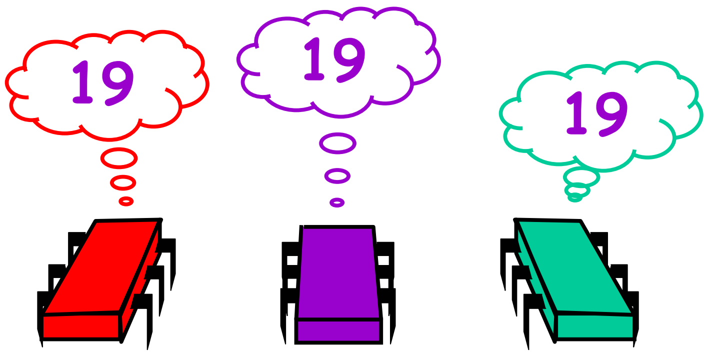
Consensus: Each Thread has a Private Input



They Communicate



They Agree on Some Thread's Input



Consensus is important

- With consensus, you can implement anything you can imagine…
- Examples: with consensus you can decide on a leader, implement mutual exclusion, or solve the two generals problem

You gonna learn

- In some models, consensus is possible
- In some other models, it is not
- Goal of this and next lecture: to learn whether for a given model consensus is possible or not … and prove it!

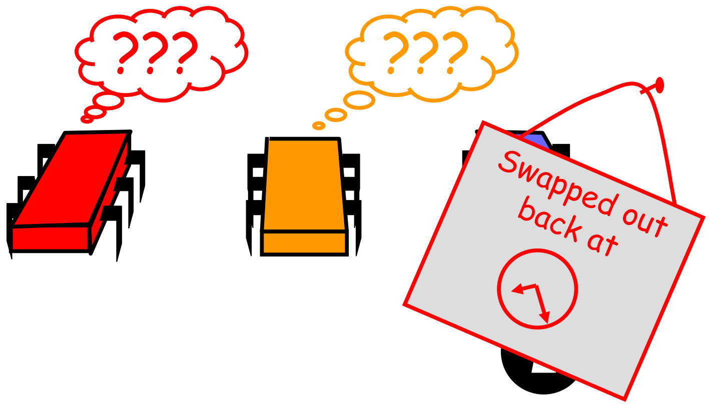
Consensus #1 shared memory

- n processors, with n > 1
- Processors can atomically *read* or *write* (not both) a shared memory cell

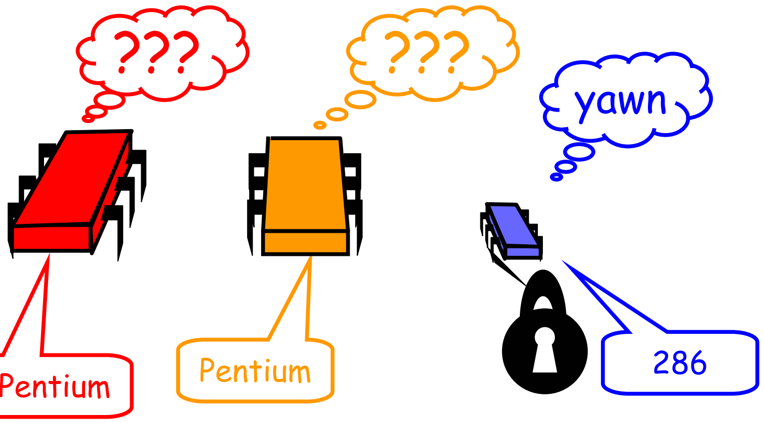
Protocol (Algorithm?)

- There is a designated memory cell c.
- Initially c is in a special state "?"
- Processor 1 writes its value v_1 into c, then decides on v_1.
- A processor j (j not 1) reads c until j reads something else than "?", and then decides on that.

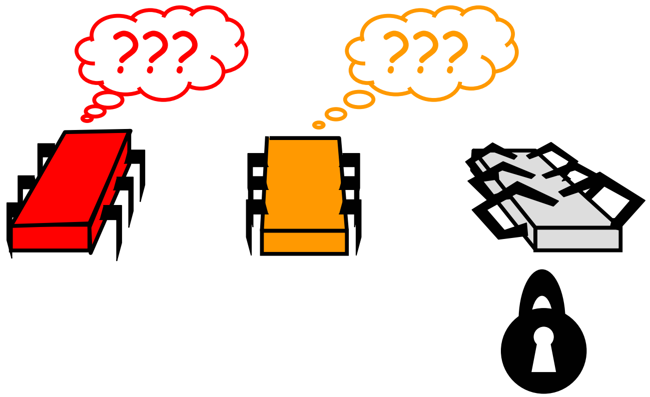
Unexpected Delay



Heterogeneous Architectures



Fault-Tolerance



Consensus #2
wait-free shared memory

- n processors, with n > 1
- Processors can atomically *read* or *write* (not both) a shared memory cell
- Processors might crash (halt)
- Wait-free implementation… huh?

Wait-Free Implementation

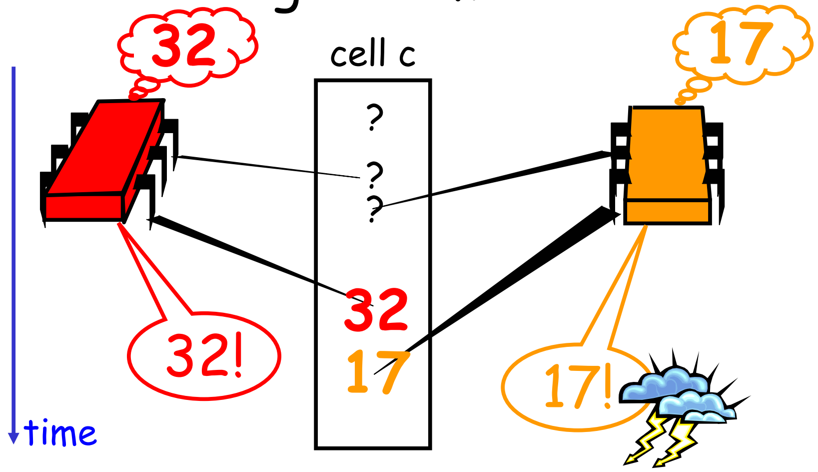
- Every process (method call) completes in a finite number of steps
- Implies no mutual exclusion
- We assume that we have wait-free atomic registers (that is, reads and writes to same register do not overlap)

A wait-free algorithm…

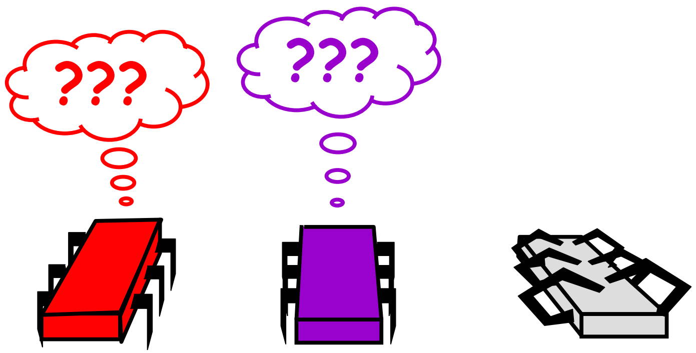
- There is a cell c, initially c="?"
- Every processor i does the following

```
r = Read(c);
if (r == "?") then
      Write(c, v_i); decide v_i;
else
      decide r;
```


Is the algorithm correct?



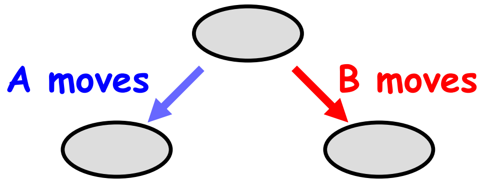
Theorem: No wait-free consensus



Proof Strategy

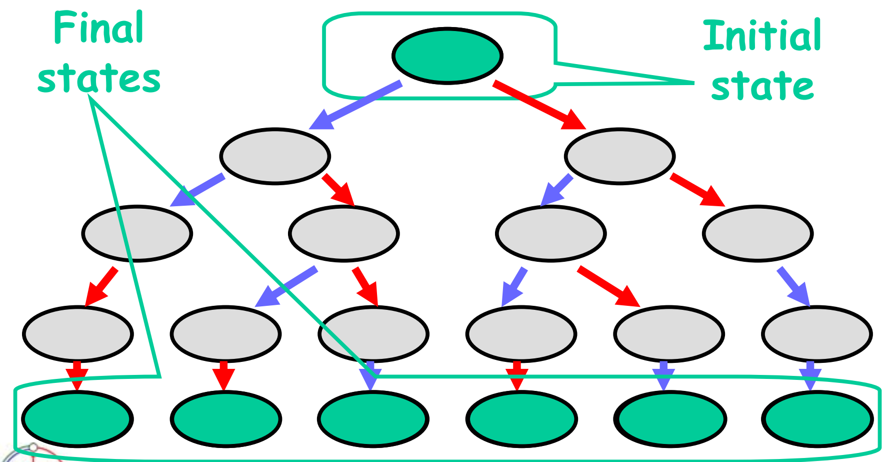
- Make it simple
  - n = 2, binary input
- Assume that there is a protocol
- Reason about the properties of any such protocol
- Derive a contradiction

Wait-Free Computation

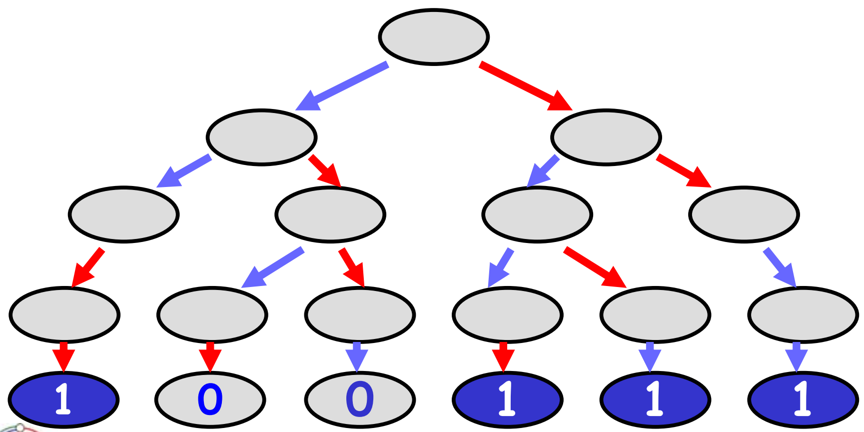


- Either A or B "moves"
- Moving means
  - Register read
  - Register write

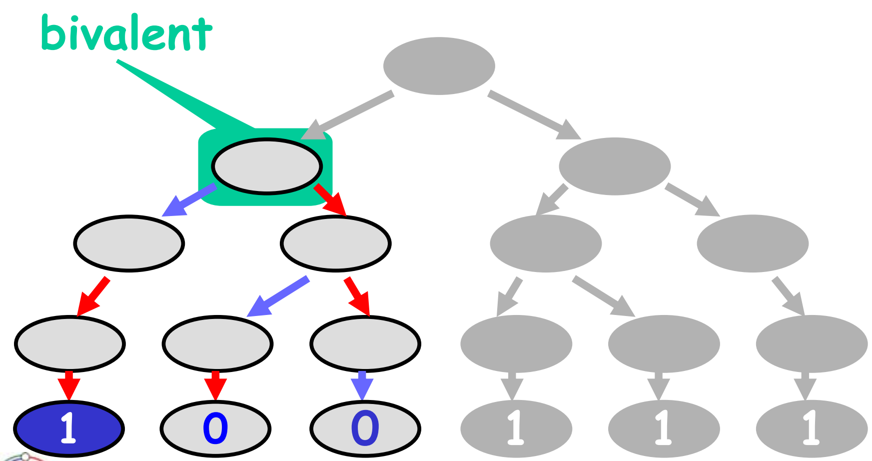
The Two-Move Tree



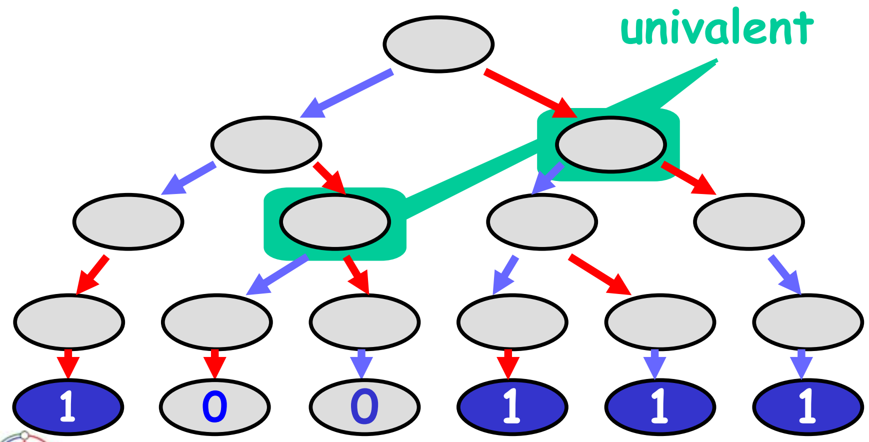
Decision Values



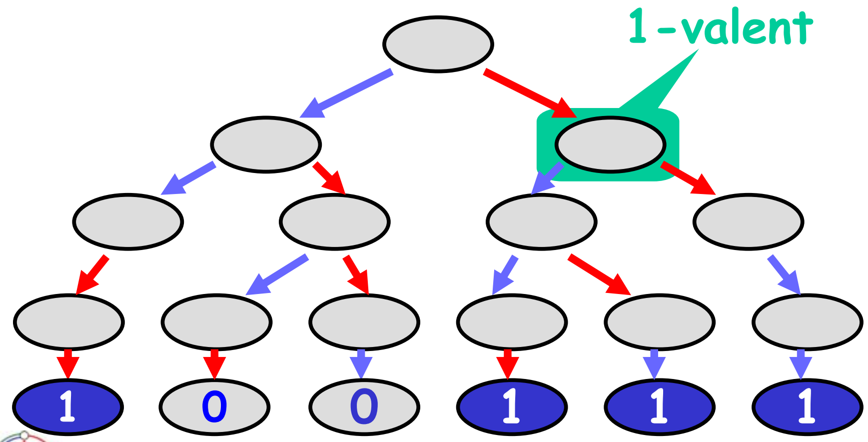
Bivalent: Both Possible



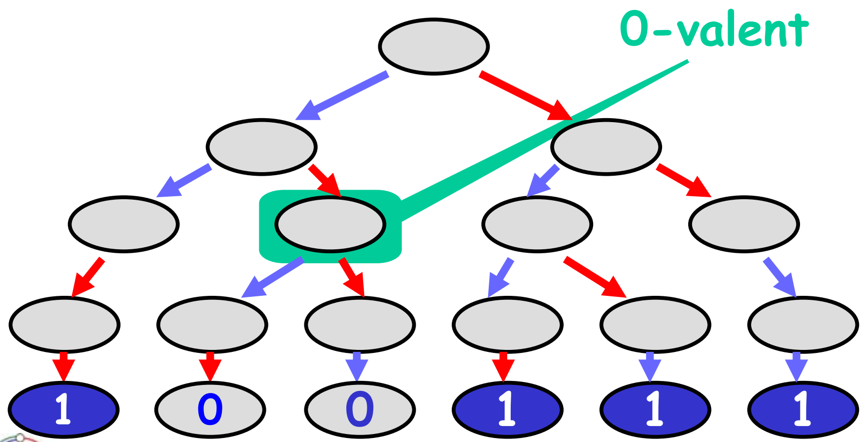
Univalent: Single Value Possible



1-valent: Only 1 Possible



0-valent: Only 0 possible



Summary

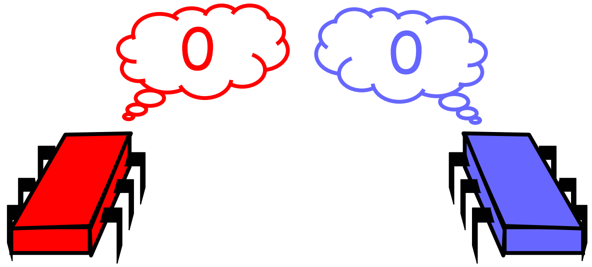
- Wait-free computation is a tree
- Bivalent system states
  - Outcome not fixed
- Univalent states
  - Outcome is fixed
  - May not be "known" yet
  - 1-Valent and 0-Valent states

Claim

Some initial system state is bivalent

(The outcome is not always fixed from the start.)

A 0-Valent Initial State



- All executions lead to decision of 0

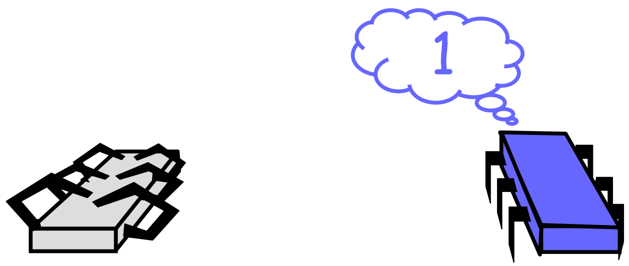
A 0-Valent Initial State

- Solo execution by A also decides 0

A 1-Valent Initial State

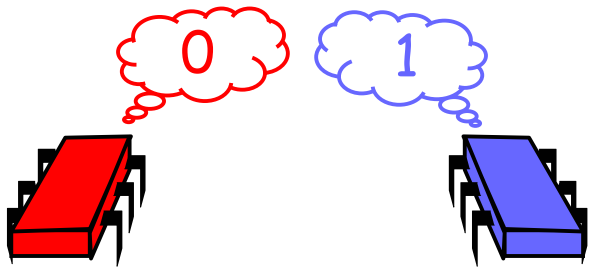
- All executions lead to decision of 1

A 1-Valent Initial State



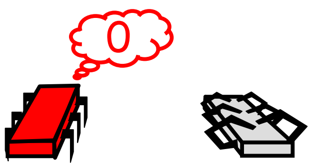
- Solo execution by B also decides 1

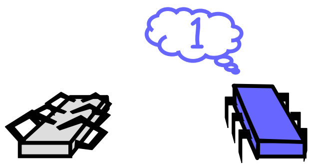
A Univalent Initial State?



- Can all executions lead to the same decision?

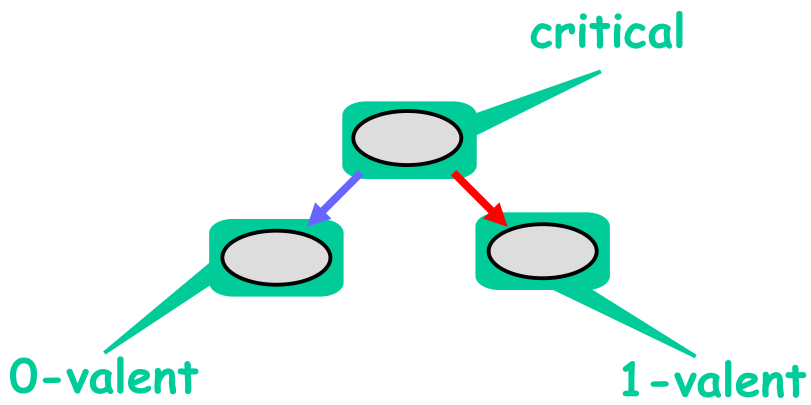
State is Bivalent





- Solo execution by A must decide 0
- Solo execution by B must decide 1

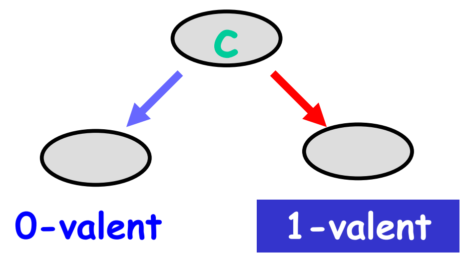
Critical States



Critical States

- Starting from a bivalent initial state
- The protocol can reach a critical state
  - Otherwise we could stay bivalent forever
  - And the protocol is not wait-free

From a Critical State



If A goes first, protocol decides 0

If B goes first, protocol decides 1

Model Dependency

- So far, memory-independent!
- True for
  - Registers
  - Message-passing
  - Carrier pigeons
  - Any kind of asynchronous computation

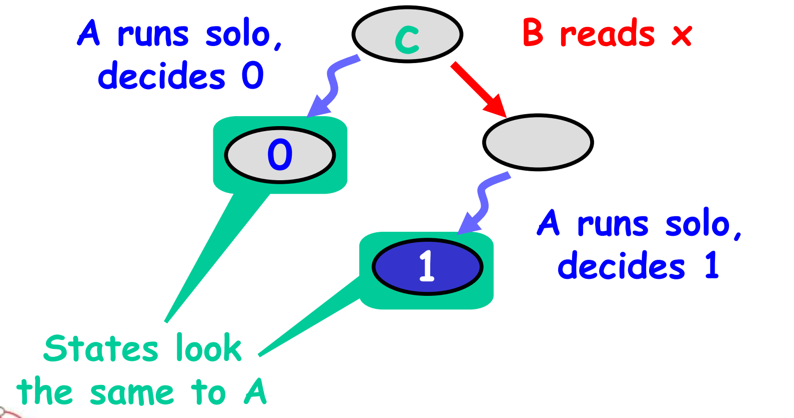
What are the Threads Doing?

- Reads and/or writes
- To same/different registers

Possible Interactions

	x.read()	y.read()	x.write()	y.write()
x.read()	?	?	?	?
y.read()	?	?	?	?
x.write()	?	?	?	?
y.write()	?	?	?	?

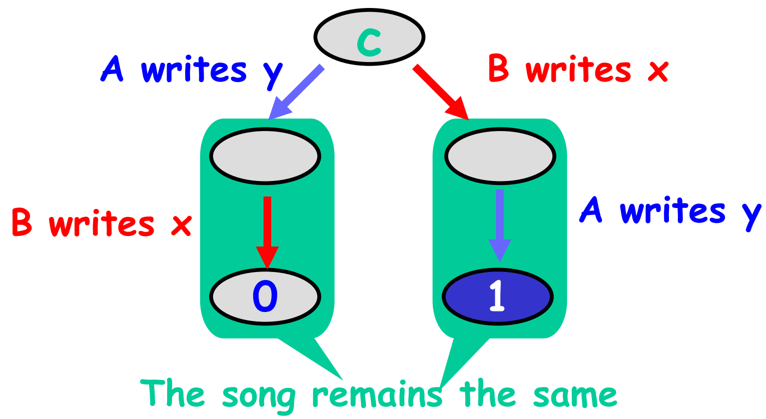
Reading Registers



Possible Interactions

	x.read()	y.read()	x.write()	y.write()
x.read()	no	no	no	no
y.read()	no	no	no	no
x.write()	no	no	?	?
y.write()	no	no	?	?

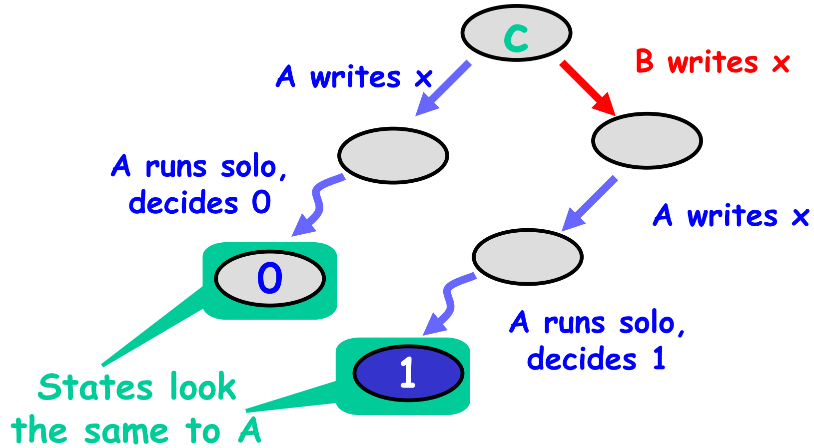
Writing Distinct Registers



Possible Interactions

	x.read()	y.read()	x.write()	y.write()
x.read()	no	no	no	no
y.read()	no	no	no	no
x.write()	no	no	?	no
y.write()	no	no	no	?

Writing Same Registers



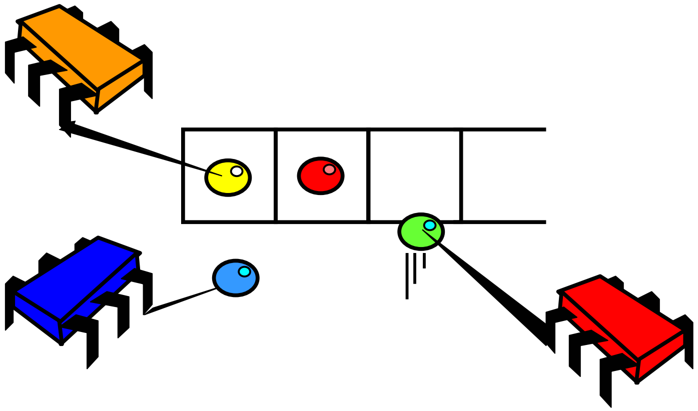
That's All, Folks!

	x.read()	y.read()	x.write()	y.write()
x.read()	no	no	no	no
y.read()	no	no	no	no
x.write()	no	no	no	no
y.write()	no	no	no	no

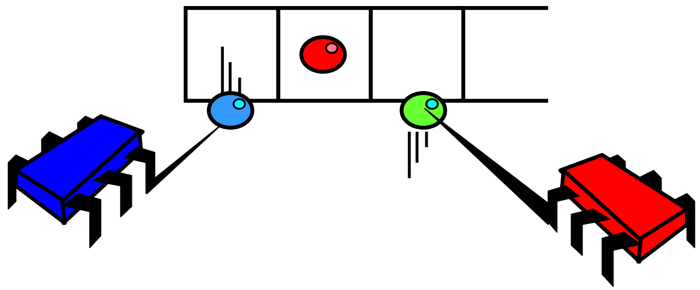
Theorem

- It is impossible to solve consensus using read/write atomic registers
  - Assume protocol exists
  - It has a bivalent initial state
  - Must be able to reach a critical state
  - Case analysis of interactions
    - Reads vs others
    - Writes vs writes

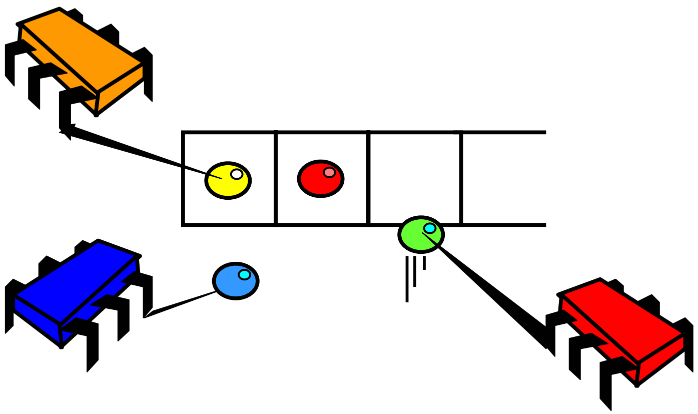
What Does Consensus have to do with Distributed Systems?



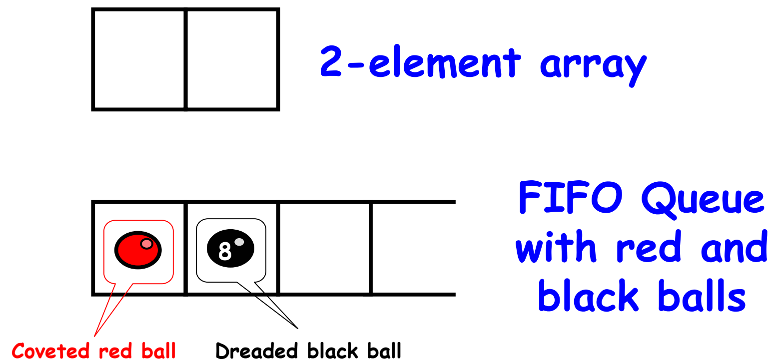
We want to build a Concurrent FIFO Queue



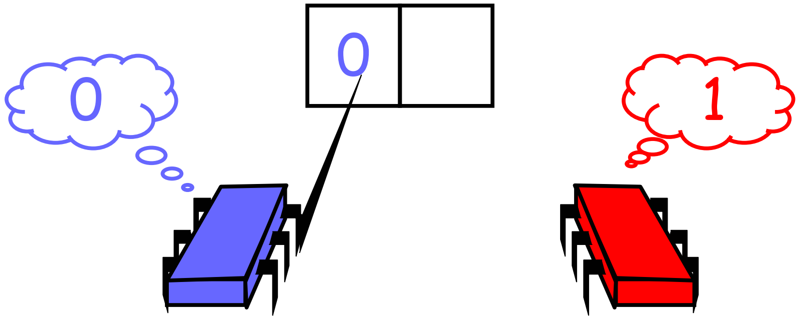
With Multiple Dequeuers!



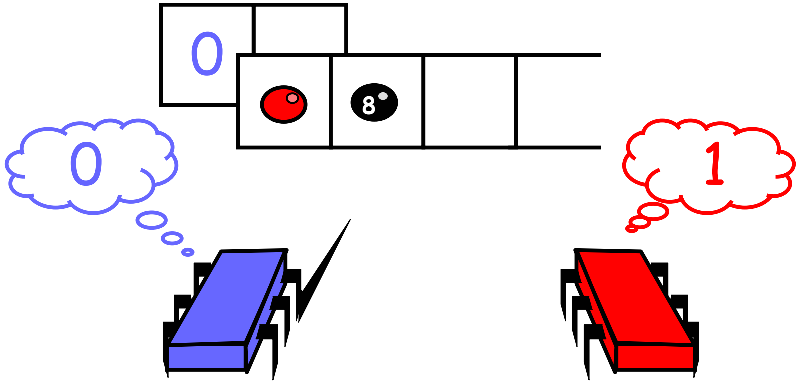
A Consensus Protocol



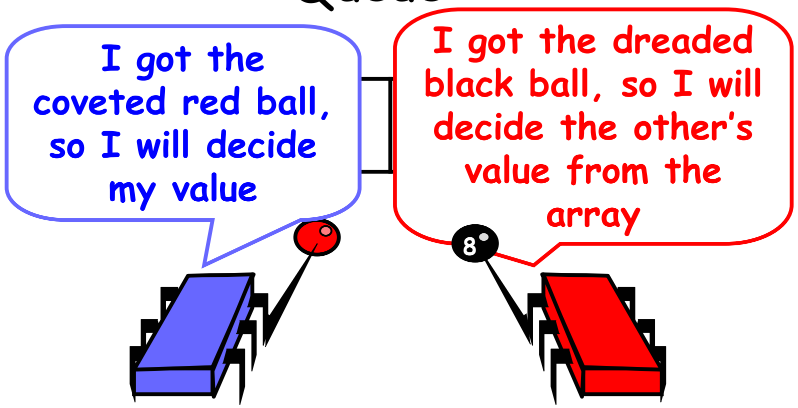
Protocol: Write Value to Array



Protocol: Take Next Item from Queue



Protocol: Take Next Item from Queue



Why does this Work?

- If one thread gets the red ball
- Then the other gets the black ball
- Winner can take her own value
- Loser can find winner's value in array
  - Because threads write array before dequeuing from queue

Implication

- We can solve 2-thread consensus using only
  - A two-dequeuer queue
  - Atomic registers

Implications

- Assume there exists
  - A queue implementation from atomic registers
- Given
  - A consensus protocol from queue and registers
- Substitution yields
  - A wait-free consensus protocol from atomic registers

contradiction

Corollary

- It is impossible to implement a two-dequeuer wait-free FIFO queue with read/write shared memory.
- This was a proof by reduction; important beyond NP-completeness…

Consensus #3 read-modify-write shared mem.

- n processors, with $n > 1$
- Wait-free implementation
- Processors can atomically read *and* write a shared memory cell in one atomic step: the value written can depend on the value read
- We call this a RMW register

Protocol

- There is a cell c, initially c="?"
- Every processor i does the following

RMW(c), with

```
if (c == "?") then
    write(c, vi); decide vi;
else
    decide c;
```

atomic step

Discussion

- Protocol works correctly
  - One processor accesses c as the first; this processor will determine decision
- Protocol is wait-free
- RMW is quite a strong primitive
  - Can we achieve the same with a weaker primitive?

Read-Modify-Write more formally

- Method takes 2 arguments:
  - Variable x
  - Function f
- Method call:
  - Returns value of x
  - Replaces x with f(x)

Read-Modify-Write

```
public abstract class RMW {
  private int value;

  public void rmw(function f) {
    int prior  = this.value;
    this.value = f(this.value);
    return prior;
  }

}
```

Return prior value

Apply function

Example: Read

```
public abstract class RMW {
  private int value;

  public void read() {
    int prior  = this.value;
    this.value = this.value;
    return prior;
  }

}
```

identity function

Example: test&set

```
public abstract class RMW {
  private int value;

  public void TAS() {
    int prior  = this.value;
    this.value = 1;
    return prior;
  }

}
```

constant function

Example: fetch&inc

```
public abstract class RMW {
  private int value;

  public void fai() {
    int prior  = this.value;
    this.value = this.value+1;
    return prior;
  }

}
```

increment function

Example: fetch&add

```
public abstract class RMW {
  private int value;

  public void faa(int x) {
    int prior  = this.value;
    this.value = this.value+x;
    return prior;
  }

}
```

addition function

Example: swap

```
public abstract class RMW {
  private int value;

  public void swap(int x) {
    int prior  = this.value;
    this.value = x;
    return prior;
  }

}
```

constant function

Example: compare&swap

```
public abstract class RMW {
  private int value;

  public void CAS(int old, int new) {
    int prior  = this.value;
    if (this.value == old)
      this.value = new;
    return prior;
  }

}
```

complex function

"Non-trivial" RMW

- Not simply read
- But
  - test&set, fetch&inc, fetch&add, swap, compare&swap, general RMW
- Definition: A RMW is non-trivial if there exists a value v such that v ≠ f(v)

Consensus Numbers (Herlihy)

- An object has consensus number n
  - If it can be used
    - Together with atomic read/write registers
  - To implement n-thread consensus
    - But not (n+1)-thread consensus

Consensus Numbers

- Theorem
  - Atomic read/write registers have consensus number 1
- Proof
  - Works with 1 process
  - We have shown impossibility with 2

Consensus Numbers

- Consensus numbers are a useful way of measuring synchronization power
- Theorem
  - If you can implement X from Y
  - And X has consensus number c
  - Then Y has consensus number at least c

Synchronization Speed Limit

- Conversely
  - If X has consensus number c
  - And Y has consensus number d < c
  - Then there is no way to construct a wait-free implementation of X by Y
- This theorem will be very useful
  - Unforeseen practical implications!

Theorem

- Any non-trivial RMW object has consensus number at least 2
- Implies no wait-free implementation of RMW registers from read/write registers
- Hardware RMW instructions not just a convenience

Proof

```
public class RMWConsensusFor2
    implements Consensus {
  private RMW r;

  public Object decide() {
    int i = Thread.myIndex();
    if (r.rmw(f) == v)
      return this.announce[i];
    else
      return this.announce[1-i];
  }}
```

Initialized to v

Am I first?

Yes, return my input

No, return other's input

Proof

- We have displayed
  - A two-thread consensus protocol
  - Using any non-trivial RMW object

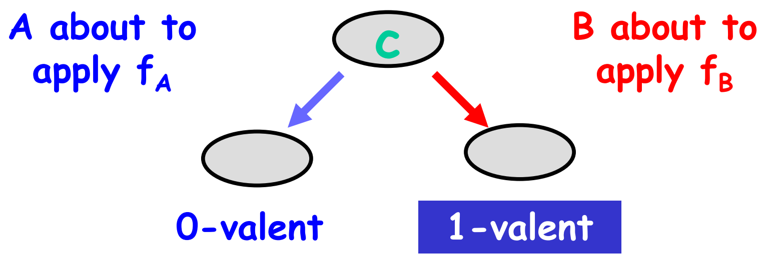
Interfering RMW

- Let F be a set of functions such that for all f_i and f_j, either
  - They commute: $f_i(f_j(x))=f_j(f_i(x))$
  - They overwrite: $f_i(f_j(x))=f_i(x)$
- Claim: Any such set of RMW objects has consensus number exactly 2

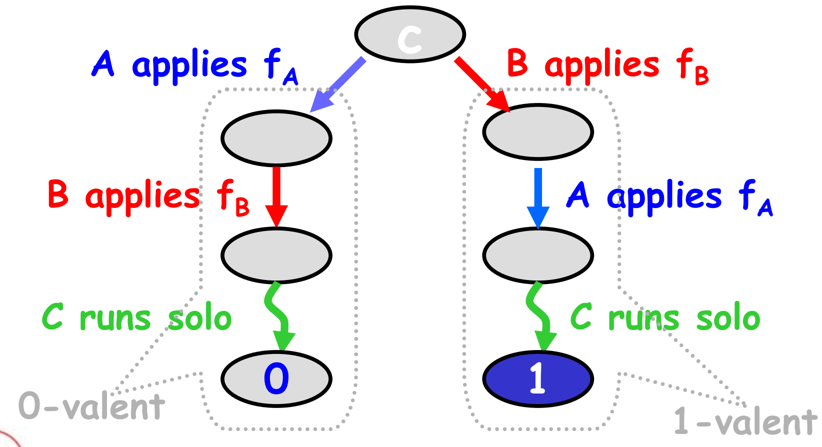
Examples

- Test-and-Set
  - Overwrite
- Swap
  - Overwrite
- Fetch-and-inc
  - Commute

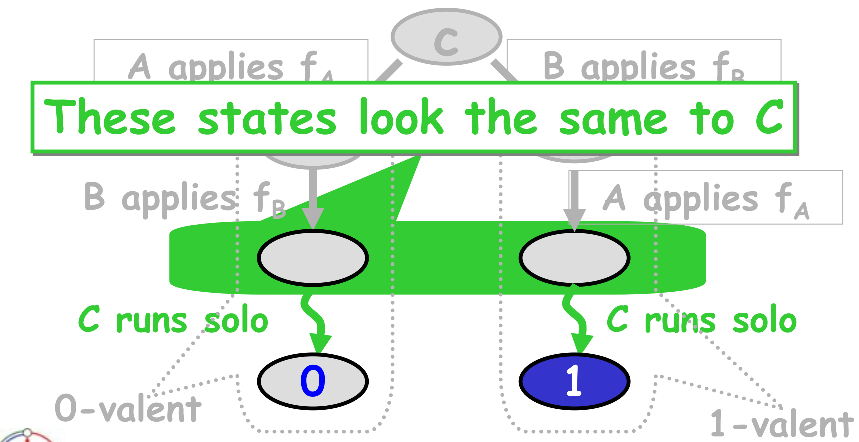
Meanwhile Back at the Critical State



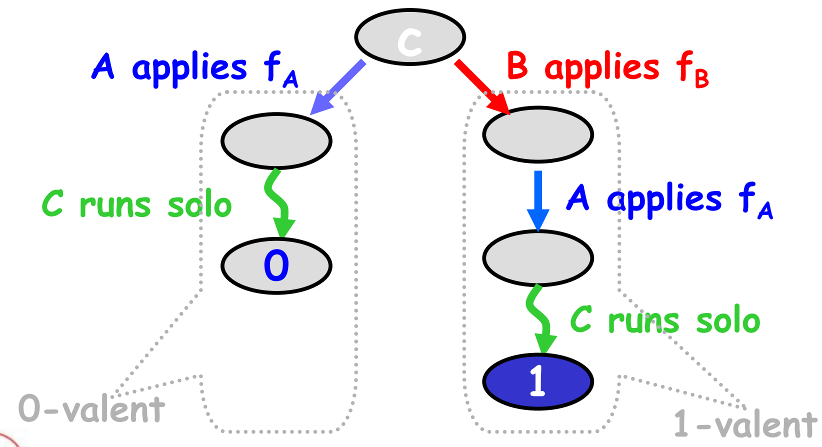
Maybe the Functions Commute



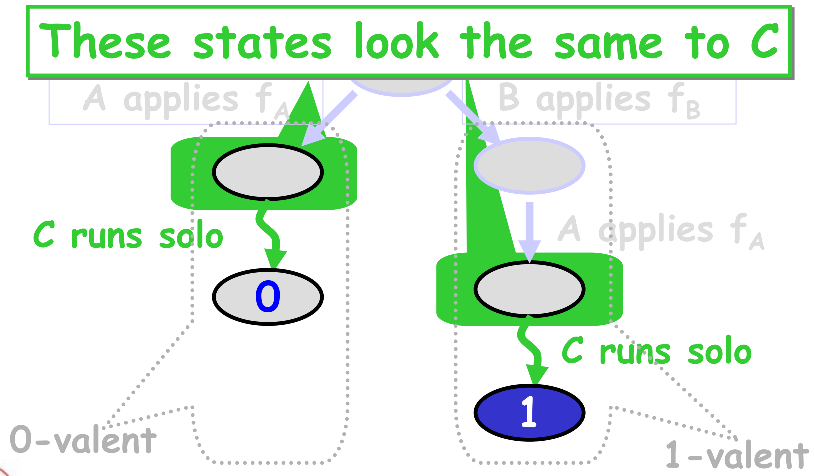
Maybe the Functions Commute



Maybe the Functions Overwrite



Maybe the Functions Overwrite



Impact

- Many early machines used these "weak" RMW instructions
  - Test-and-set (IBM 360)
  - Fetch-and-add (NYU Ultracomputer)
  - Swap
- We now understand their limitations
  - But why do we want consensus anyway?

CAS has Unbounded Consensus Number

```
public class RMWConsensus
    implements Consensus {
  private RMW r;

  public Object decide() {
    int i = Thread.myIndex();
    int j = r.CAS(-1,i);
    if (j == -1)
      return this.announce[i];
    else
      return this.announce[j];
}}
```

Initialized to -1

Am I first?

Yes, return my input

No, return other's input

The Consensus Hierarchy

1	Read/Write Registers, …
2	T&S, F&I, Swap, …
. . .	
∞	CAS, …

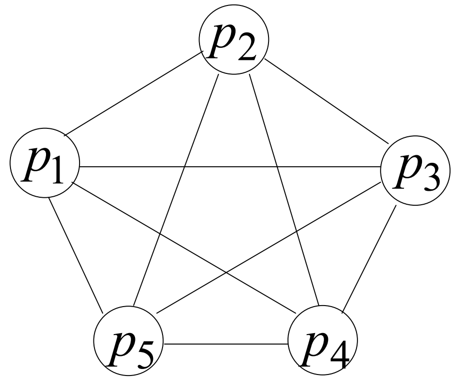
Consensus #4
Synchronous Systems

- In real systems, one can sometimes tell if a processor had crashed
  - Timeouts
  - Broken TCP connections

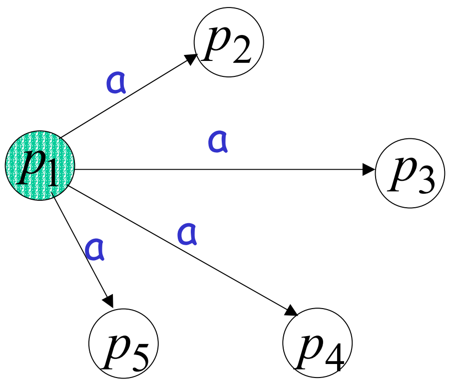
- Can one solve consensus at least in synchronous systems?

Communication Model

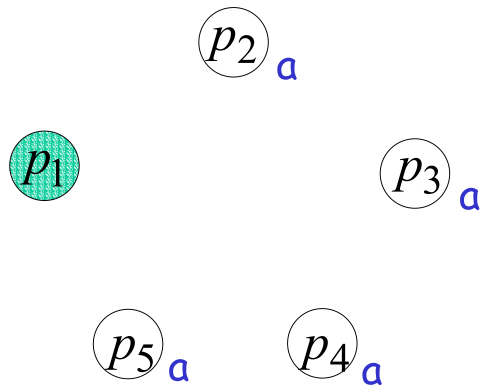
- Complete graph
- Synchronous



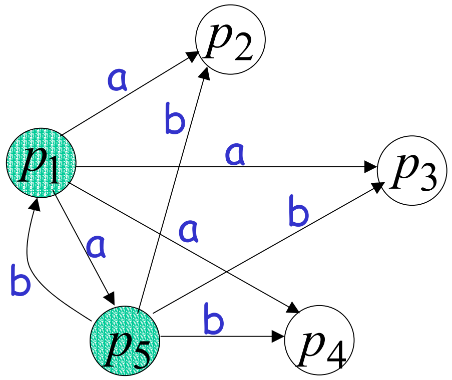
Send a message to all processors in one round: Broadcast



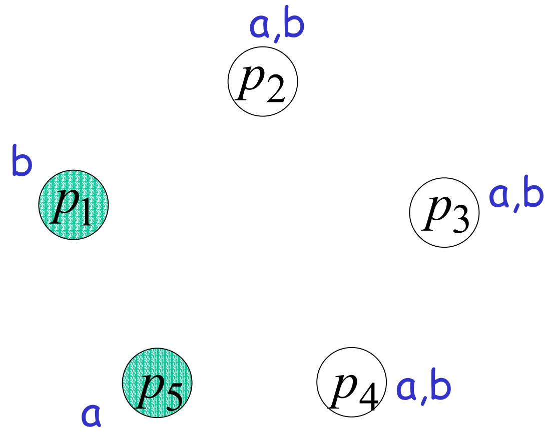
At the end of the round: everybody receives a



Broadcast: Two or more processes can broadcast in the same round

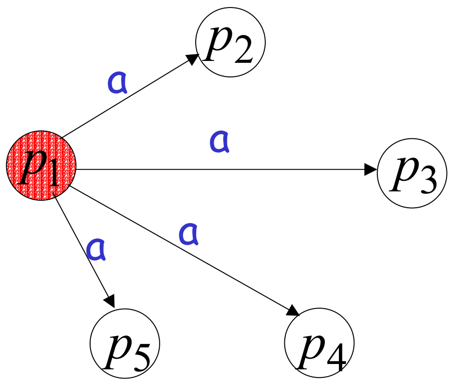


At end of round...



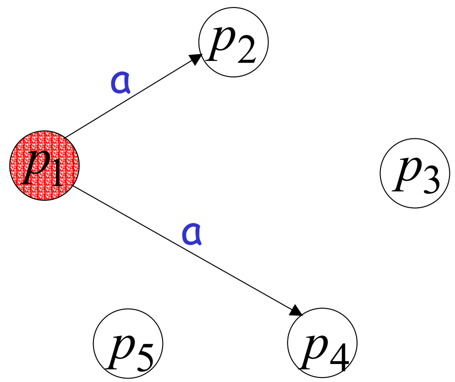
Crash Failures

Faulty processor

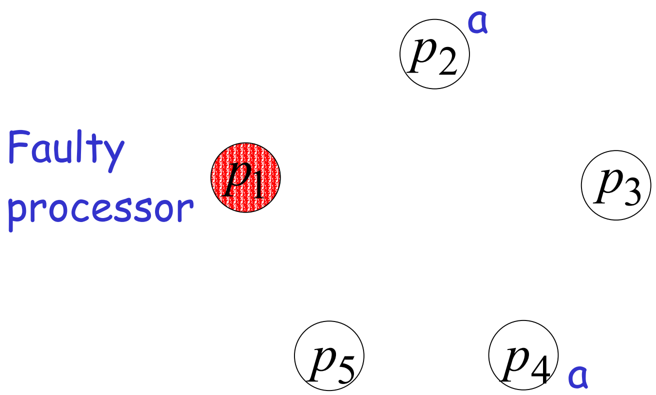


Some of the messages are lost, they are never received

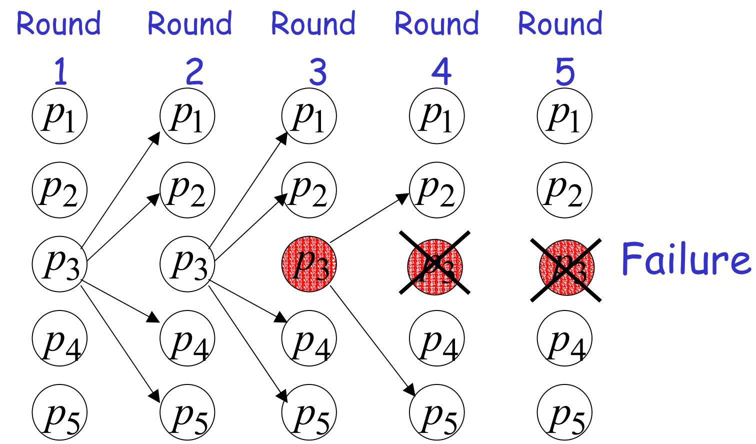
Faulty processor



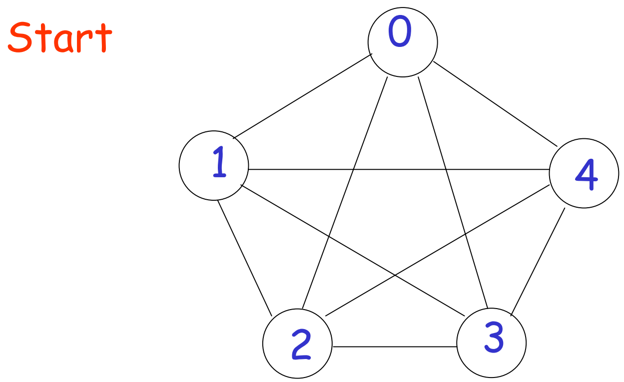
Effect



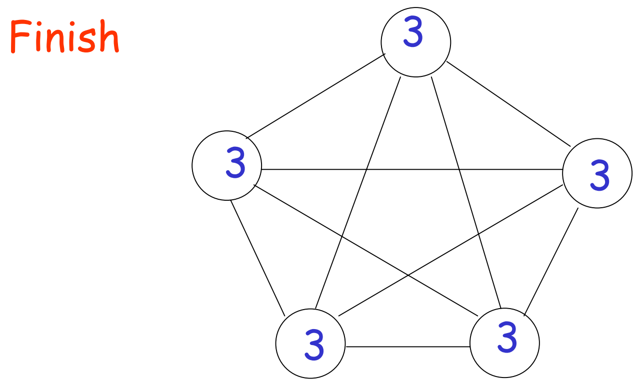
After a failure, the process disappears from the network



Consensus: Everybody has an initial value



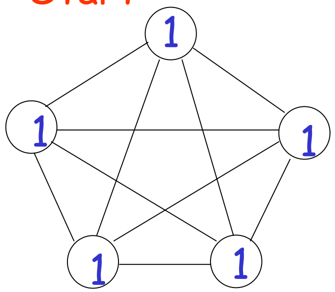
Everybody must decide on the same value



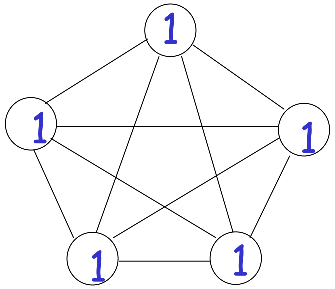
Validity condition:

If everybody starts with the same value they must decide on that value

Start



Finish



A simple algorithm

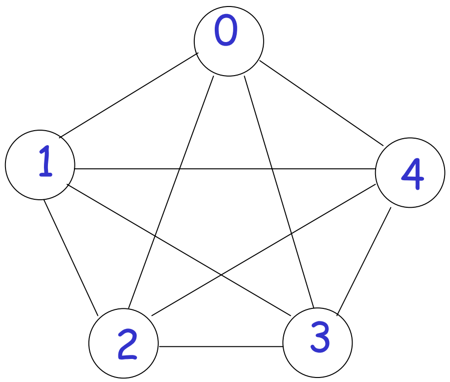
Each processor:

1. Broadcasts value to all processors

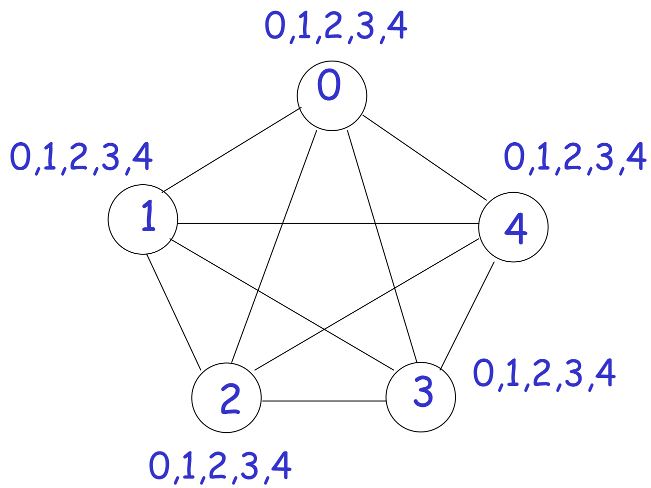
2. Decides on the minimum

(only one round is needed)

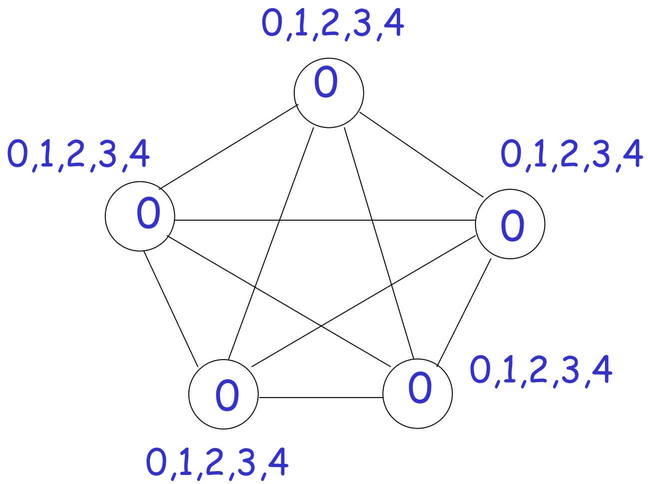
Start



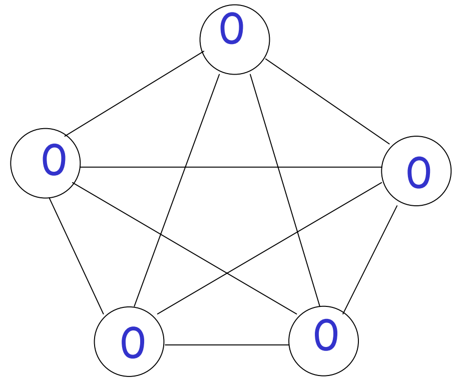
Broadcast values



Decide on minimum

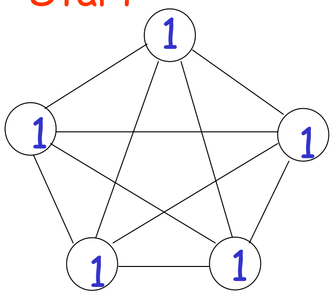


Finish

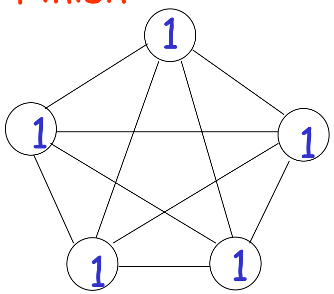


This algorithm satisfies the validity condition

Start



Finish



If everybody starts with the same initial value, everybody sticks to that value (minimum)

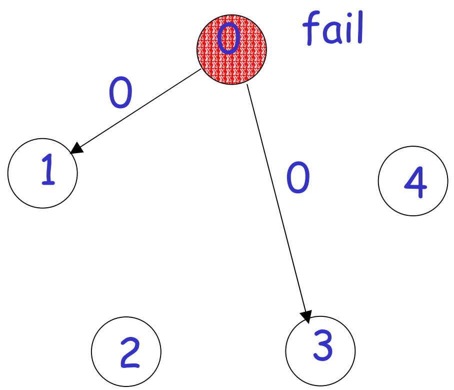
Consensus with Crash Failures

The simple algorithm doesn't work

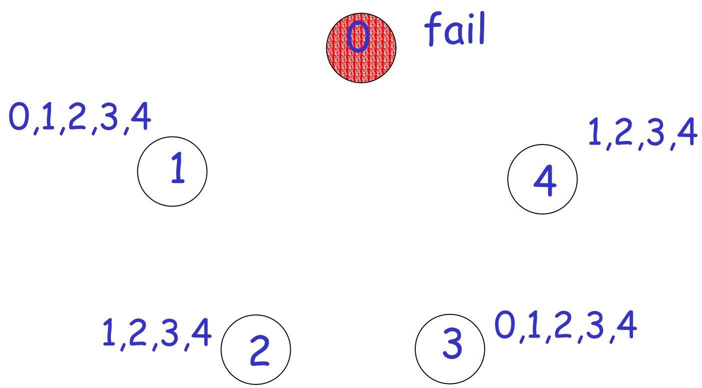
Each processor:

1. Broadcasts value to all processors
2. Decides on the minimum

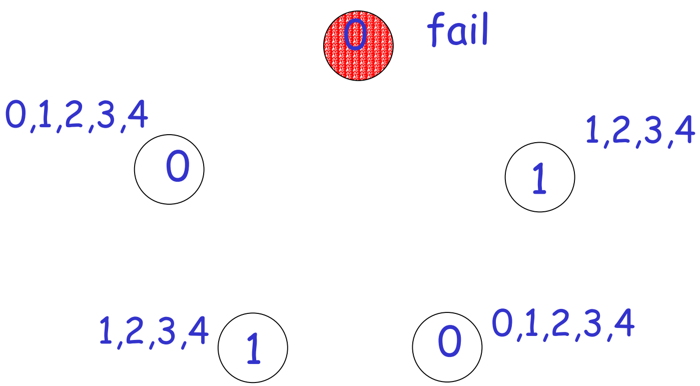
Start The failed processor doesn't broadcast its value to all processors



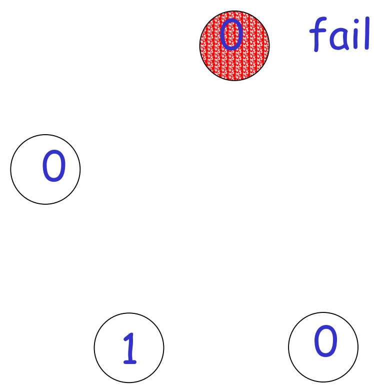
Broadcasted values



Decide on minimum



Finish - No Consensus!

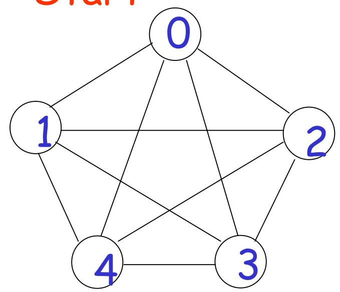


If an algorithm solves consensus for
f failed processes we say it is

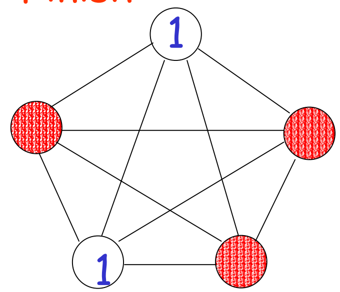
an f-resilient consensus algorithm

Example: The input and output of a 3-resilient consensus algorithm

Start



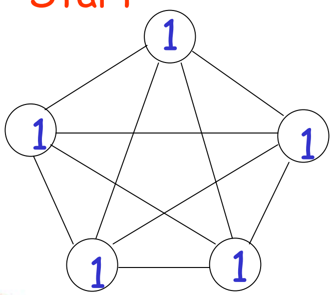
Finish



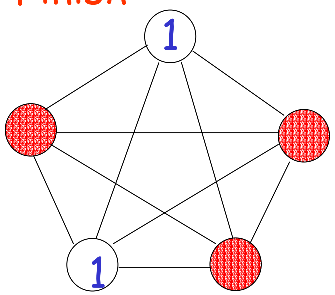
New validity condition:

if all non-faulty processes start with the same value then all non-faulty processes decide on that value

Start



Finish



An f-resilient algorithm

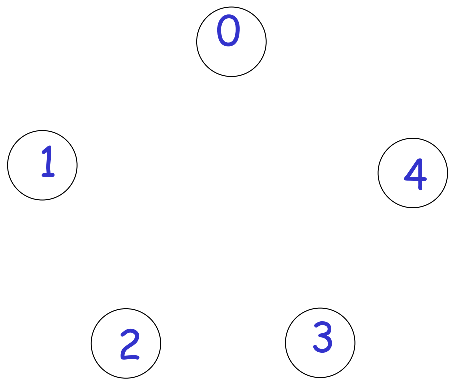
Round 1:
  Broadcast my value

Round 2 to round f+1:
  Broadcast any new received values

End of round f+1:
  Decide on the minimum value received

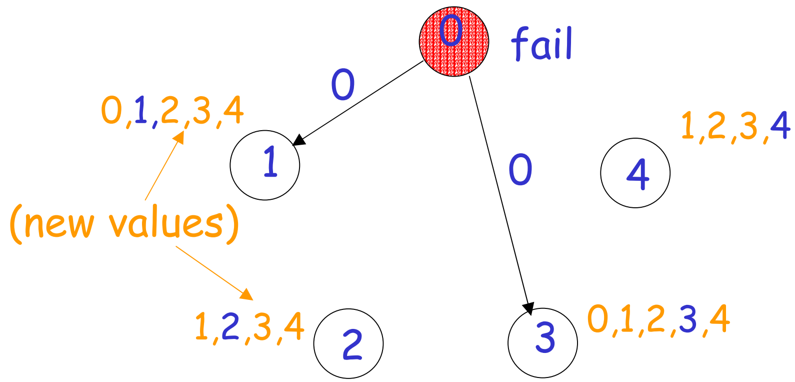
Example: f=1 failures, f+1=2 rounds needed

Start



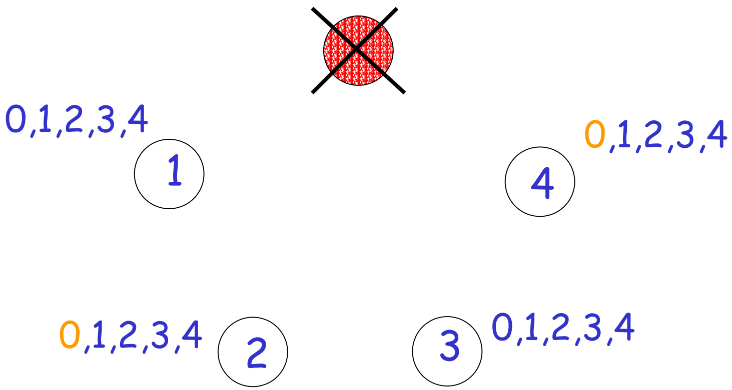
Example: f=1 failures, f+1 = 2 rounds needed

Round 1 Broadcast all values to everybody



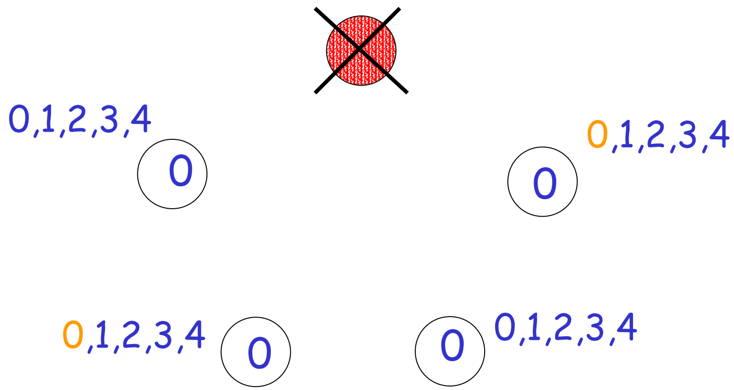
Example: f=1 failures, f+1 = 2 rounds needed

Round 2 Broadcast all new values to everybody



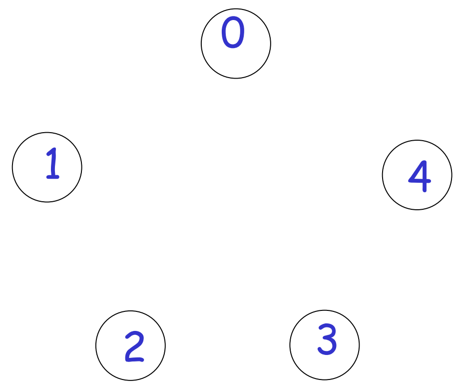
Example: f=1 failures, f+1 = 2 rounds needed

Finish Decide on minimum value



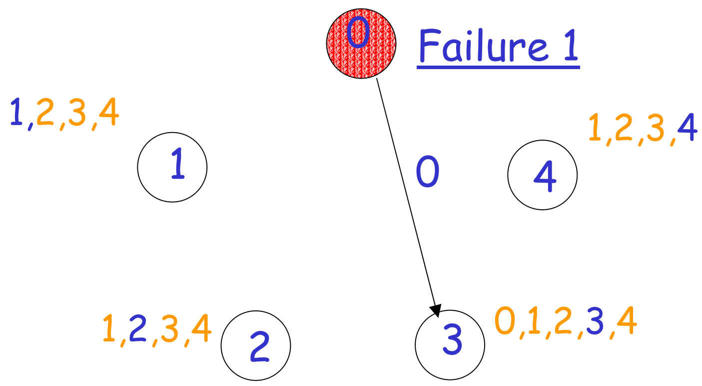
Example: f=2 failures, f+1 = 3 rounds needed

Start Example of execution with 2 failures



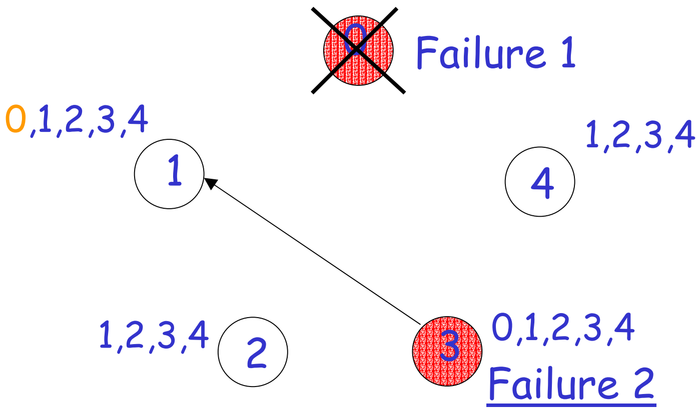
Example: f=2 failures, f+1 = 3 rounds needed

Round 1 Broadcast all values to everybody



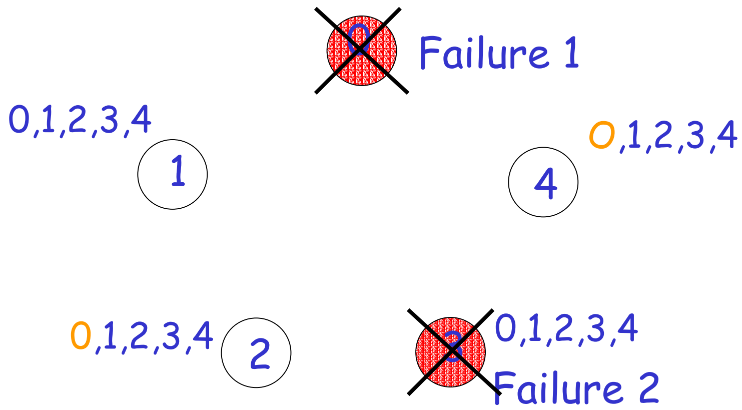
Example: f=2 failures, f+1 = 3 rounds needed

Round 2 Broadcast new values to everybody



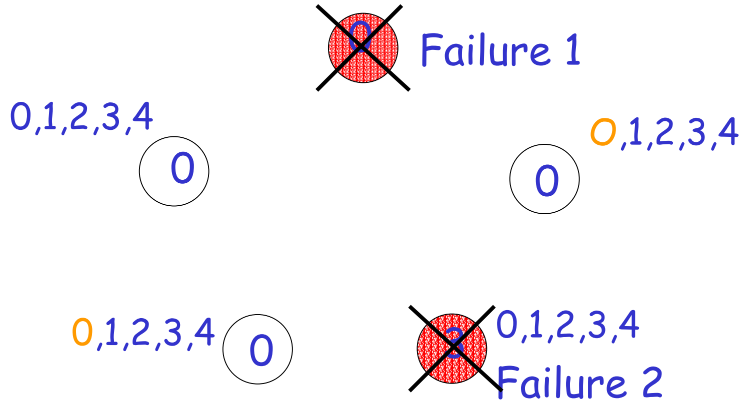
Example: f=2 failures, f+1 = 3 rounds needed

Round 3 Broadcast new values to everybody



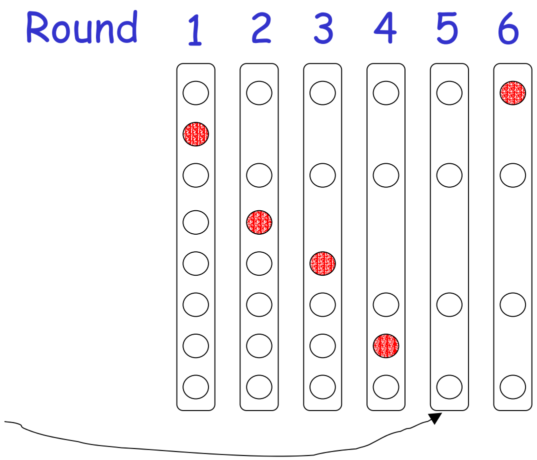
Example: f=2 failures, f+1 = 3 rounds needed

Finish Decide on the minimum value



If there are f failures and f+1 rounds then there is a round with no failed process

Example:
5 failures,
6 rounds



No failure

At the end of the round with no failure:

- Every (non faulty) process knows about all the values of all the other participating processes
- This knowledge doesn't change until the end of the algorithm

Therefore, at the end of the round with no failure:

Everybody would decide on the same value

However, as we don't know the exact position of this round,
we have to let the algorithm execute for f+1 rounds

Validity of algorithm:

when all processes start with the same input value then the consensus is that value

This holds, since the value decided from each process is some input value

A Lower Bound

Theorem: Any f-resilient consensus algorithm requires at least f+1 rounds

Proof sketch:

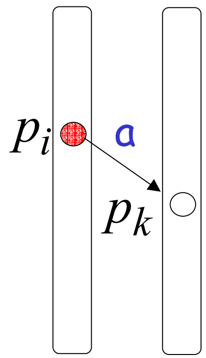
Assume for contradiction that f or less rounds are enough

Worst case scenario:

There is a process that fails in each round

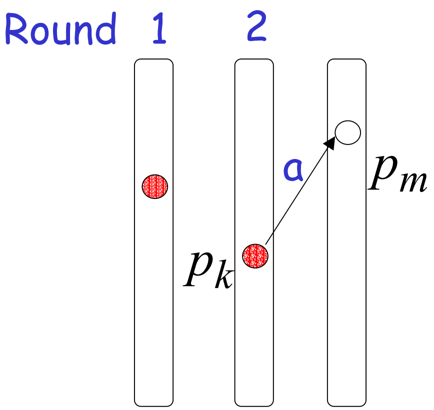
Worst case scenario

Round 1



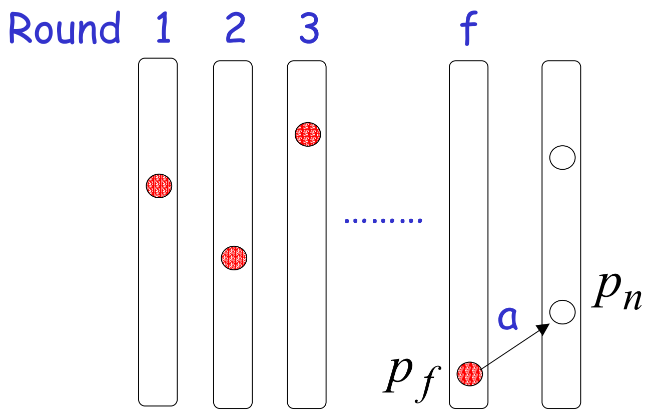
before process p_i fails, it sends its value **a** to only one process p_k

Worst case scenario



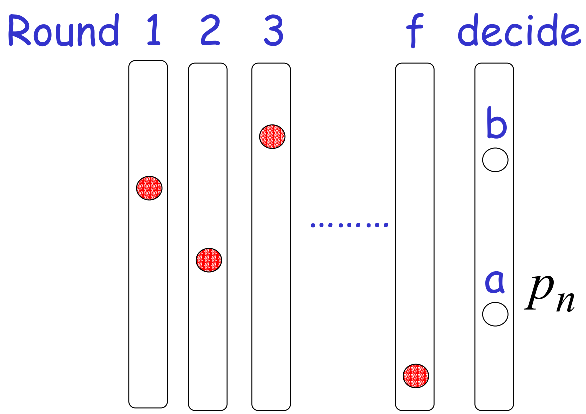
before process p_k fails, it sends value **a** to only one process p_m

Worst case scenario



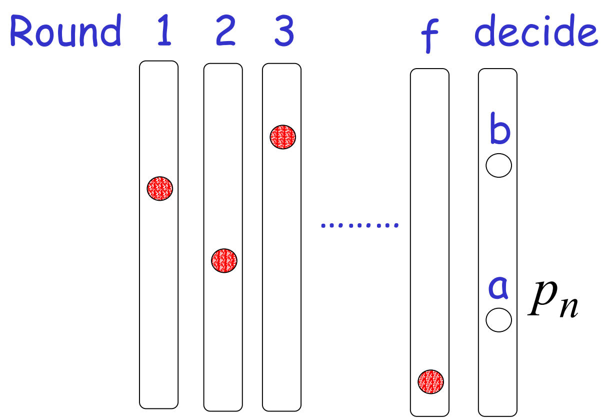
At the end of round **f** only one process p_n knows about value **a**

Worst case scenario



Process p_n may decide on **a**, and all other processes may decide on another value **(b)**

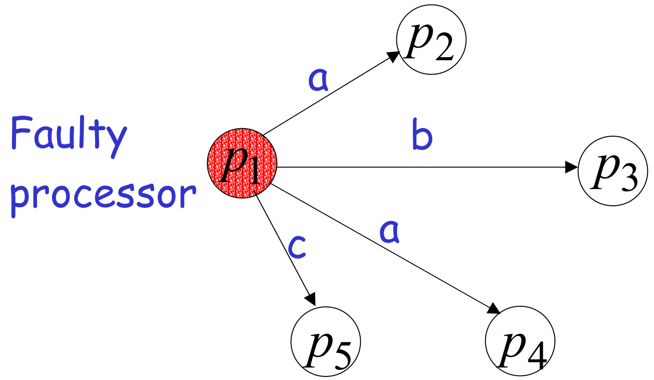
Worst case scenario



Therefore f rounds are not enough

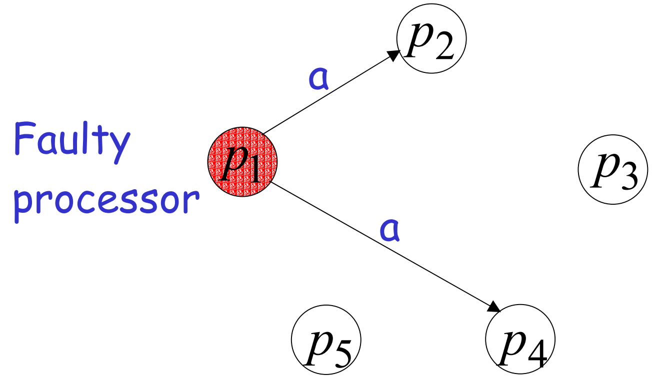
At least f+1 rounds are needed

Consensus #5
Byzantine Failures

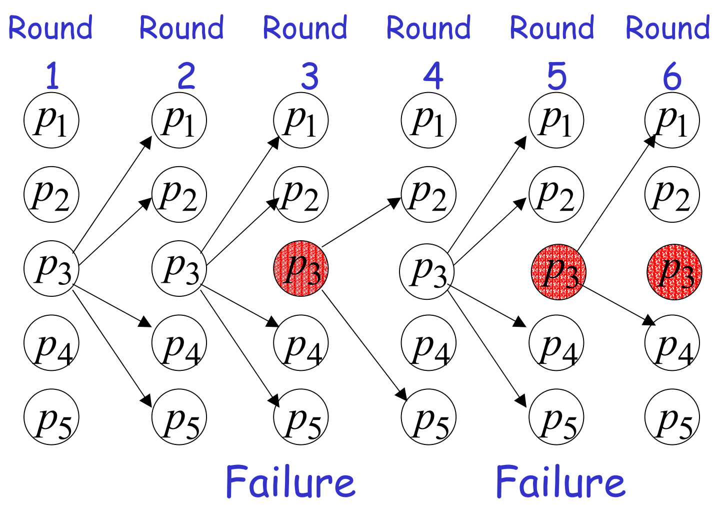


Different processes receive different values

Some messages may be lost



A Byzantine process can behave like a
Crashed-failed process



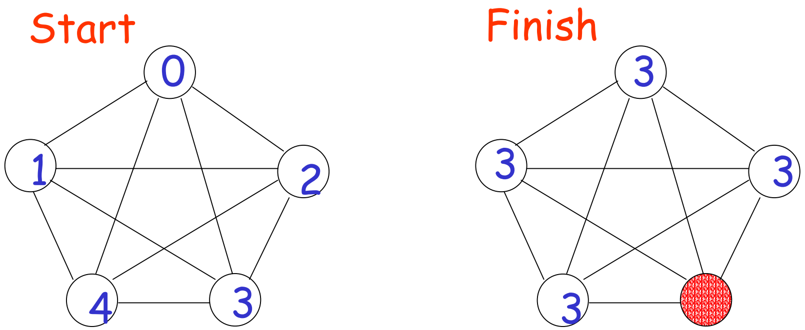
After failure the process continues
functioning in the network

Consensus with Byzantine Failures

f-resilient consensus algorithm:

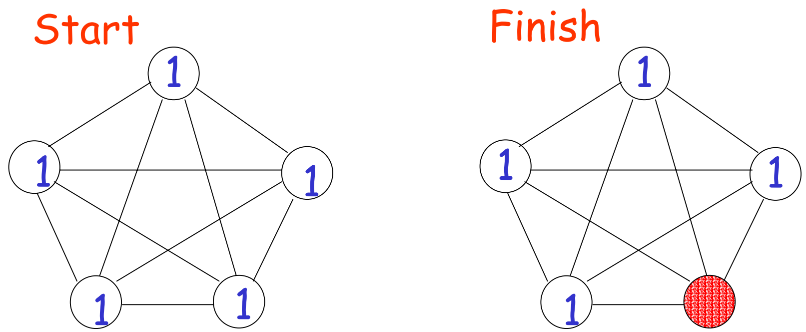
solves consensus for f failed processes

Example: The input and output of a 1-resilient consensus algorithm



Validity condition:

if all non-faulty processes start with the same value then all non-faulty processes decide on that value



Lower bound on number of rounds

Theorem: Any f-resilient consensus algorithm requires at least f+1 rounds

Proof: follows from the crash failure lower bound

Upper bound on failed processes

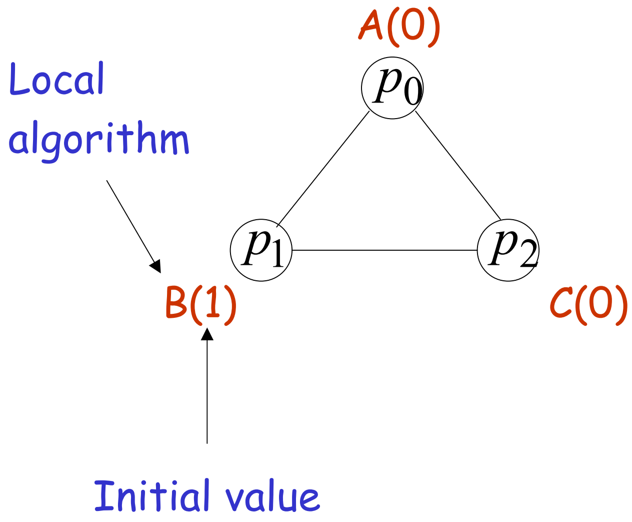
Theorem: There is no f-resilient algorithm for n processes, where $f \geq n/3$

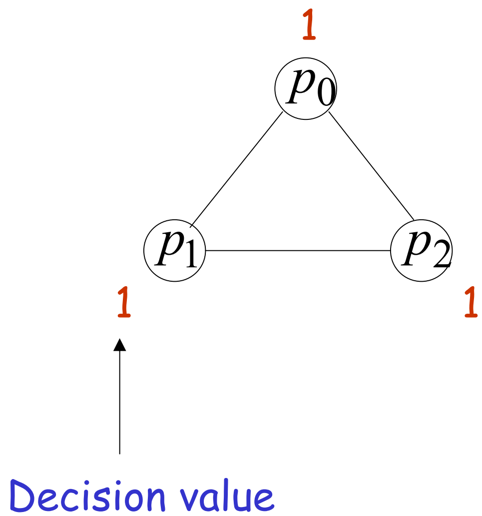
Plan: First we prove the 3 process case, and then the general case

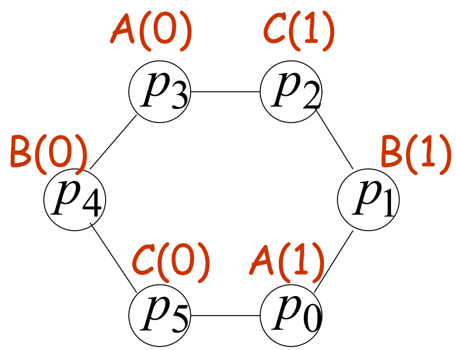
The 3 processes case

Lemma: There is no 1-resilient algorithm for 3 processes

Proof: Assume for contradiction that there is a 1-resilient algorithm for 3 processes

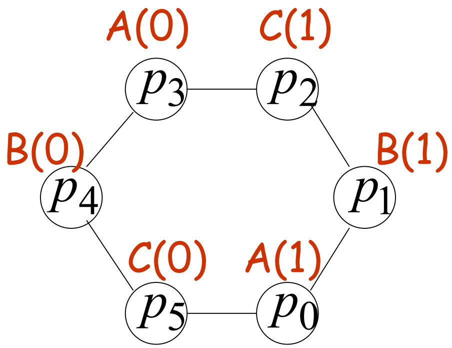


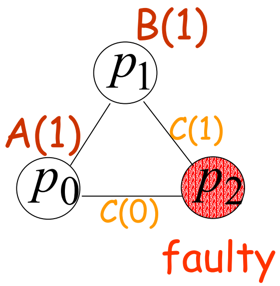




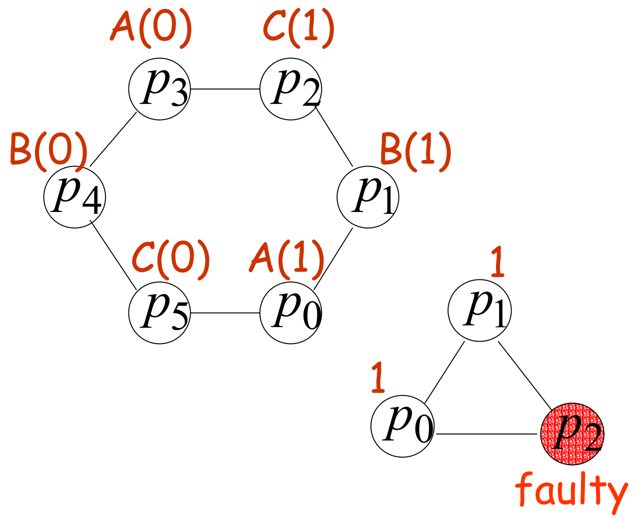
Assume 6 processes are in a ring

(just for fun)

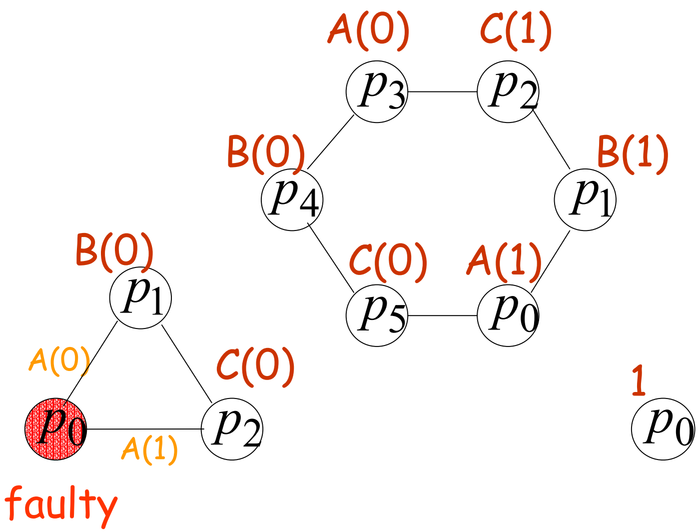


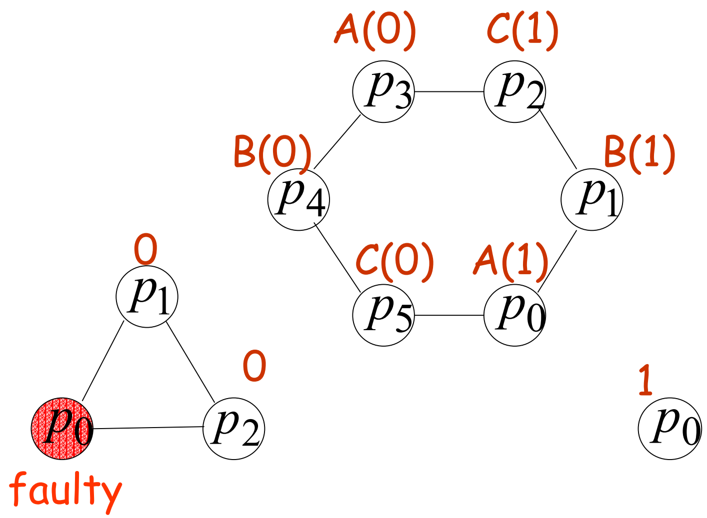


Processes think they are in a triangle

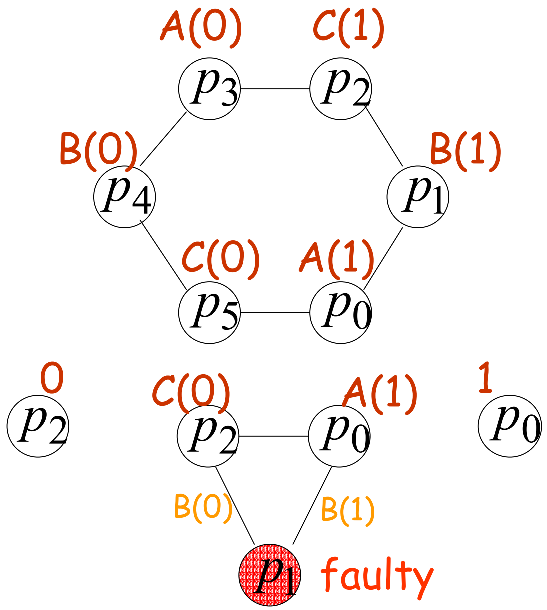


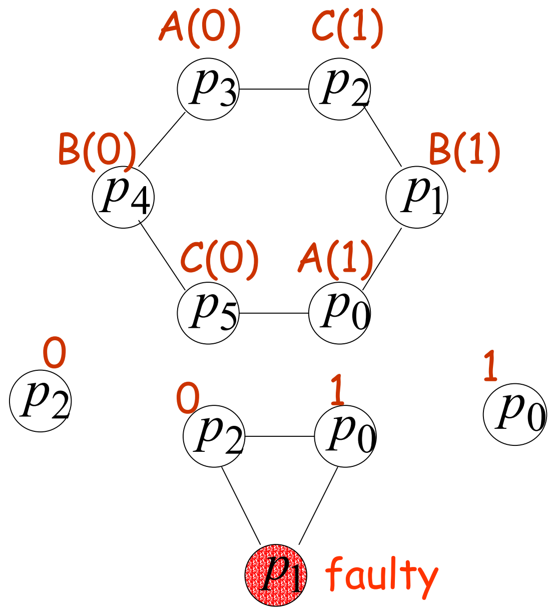
(validity condition)



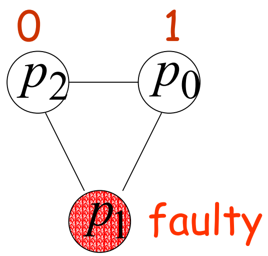


(validity condition)





Impossibility



Conclusion

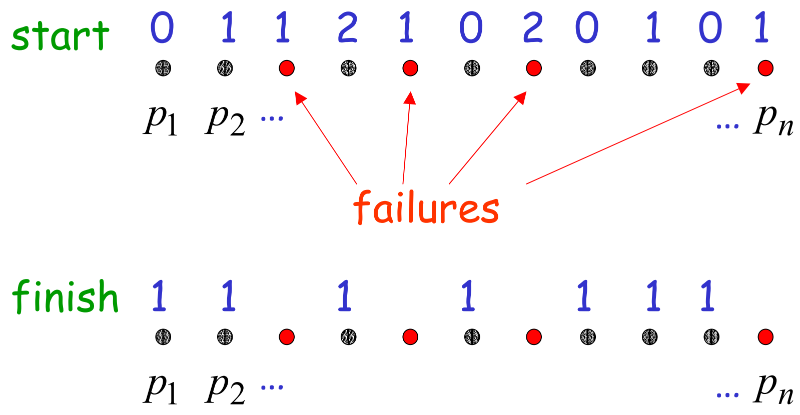
There is no algorithm that solves consensus for 3 processes in which 1 is a byzantine process

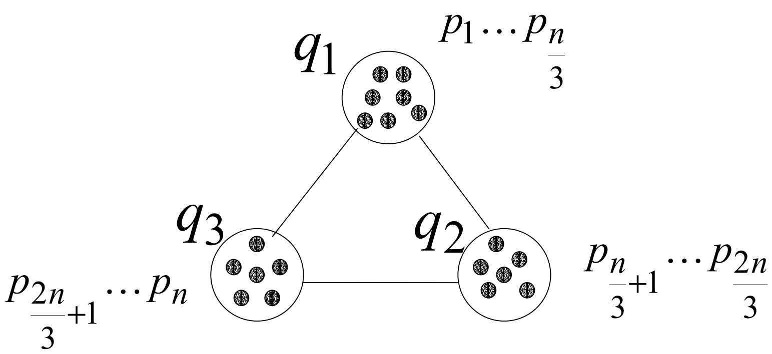
The n processes case

Assume for contradiction that there is an f-resilient algorithm A for n processes, where $f \geq n/3$

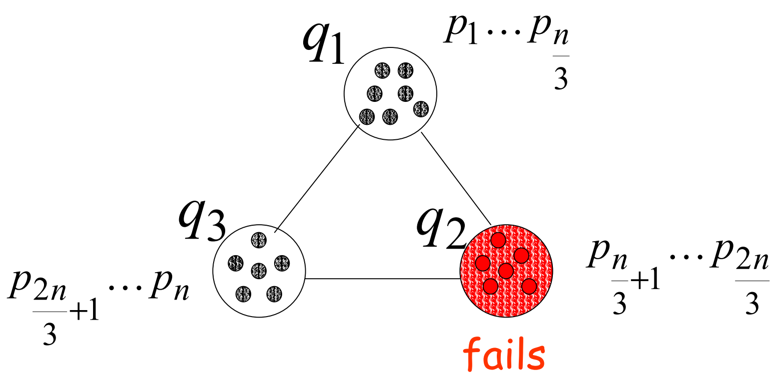
We will use algorithm A to solve consensus for 3 processes and 1 failure (which is impossible, thus we have a contradiction)

Algorithm A



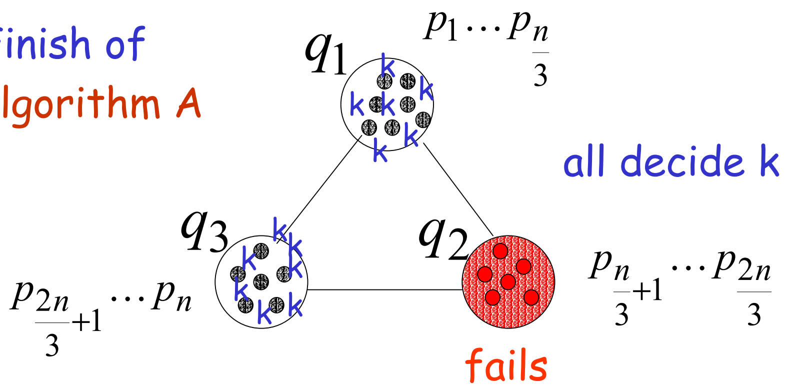


Each process q simulates algorithm A on $n/3$ of "p" processes



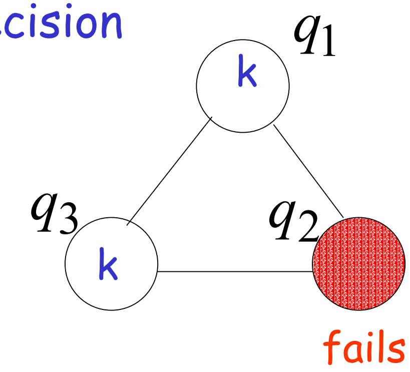
When a single q is byzantine, then $n/3$ of the "p" processes are byzantine too.

Finish of algorithm A



algorithm A tolerates $n/3$ failures

Final decision



We reached consensus with 1 failure

Impossible!!!

Conclusion

There is no f-resilient algorithm for n processes with $f \geq n/3$

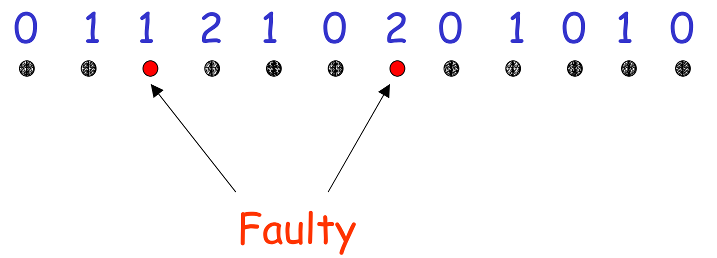
The King Algorithm

solves consensus with n processes and f failures where $f < n/4$ in $f+1$ "phases"

There are f+1 phases
Each phase has two rounds
In each phase there is a different king

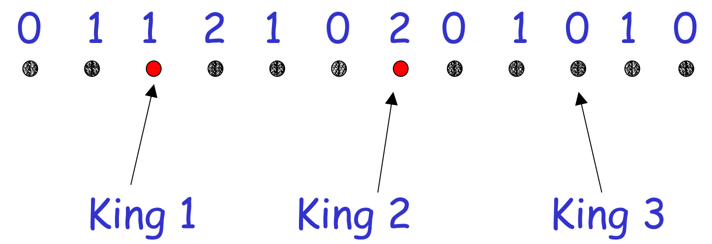
Example: 12 processes, 2 faults, 3 kings

initial values



Example: 12 processes, 2 faults, 3 kings

initial values



Remark: There is a king that is not faulty

The **King** algorithm

Each processor p_i has a preferred value v_i

In the beginning, the preferred value is set to the initial value

The **King** algorithm: Phase k

Round 1, processor p_i :

- Broadcast preferred value v_i
- Set v_i to the majority of values received

The **King** algorithm: Phase k

Round 2, king p_k :

- Broadcast new preferred value v_k

Round 2, process p_i :

- If v_i had majority of less than $\frac{n}{2}+f$

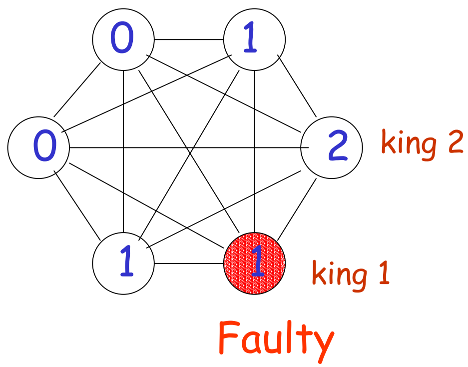
then set v_i to v_k

The **King** algorithm

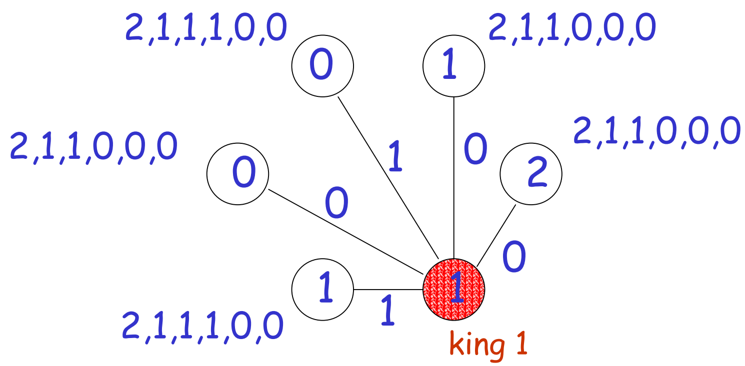
End of Phase f+1:

Each process decides on preferred value

Example: 6 processes, 1 fault



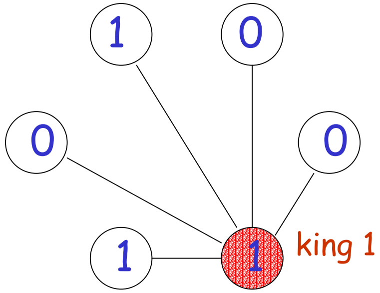
Phase 1, Round 1



Everybody broadcasts

Phase 1, Round 1

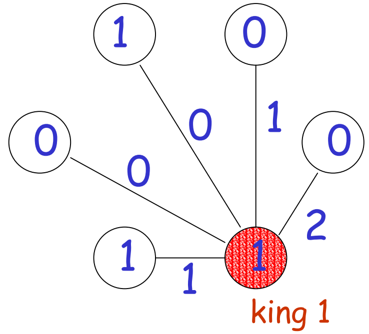
Choose the majority



Each majority population was $3 \le \frac{n}{2} + f = 4$

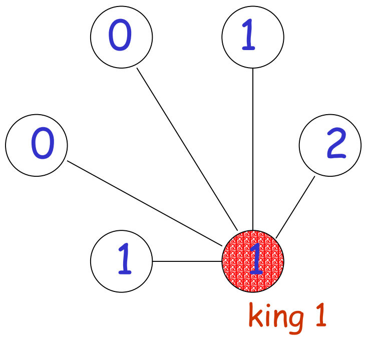
On round 2, everybody will choose the king's value

Phase 1, Round 2



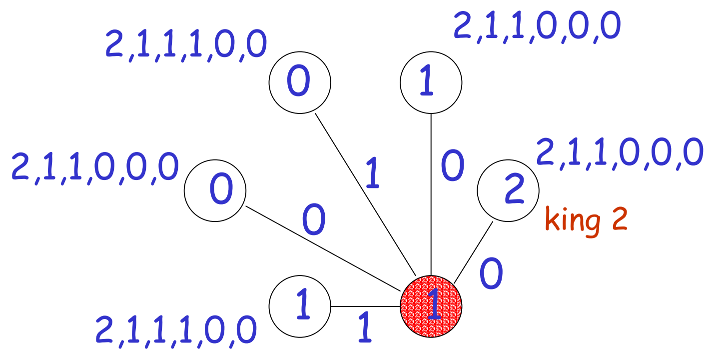
The king broadcasts

Phase 1, Round 2



Everybody chooses the king's value

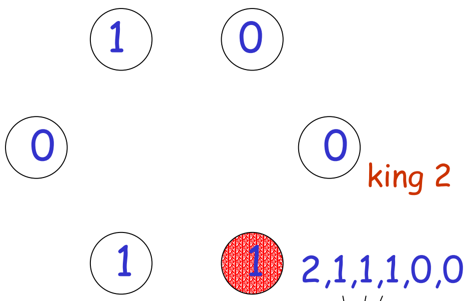
Phase 2, Round 1



Everybody broadcasts

Phase 2, Round 1

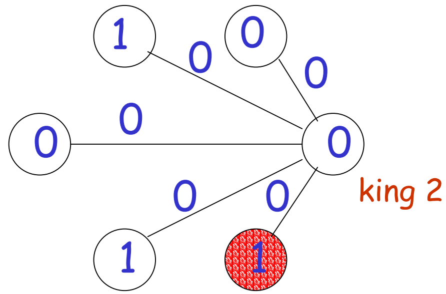
Choose the majority



Each majority population is $3 \leq \frac{n}{2} + f = 4$

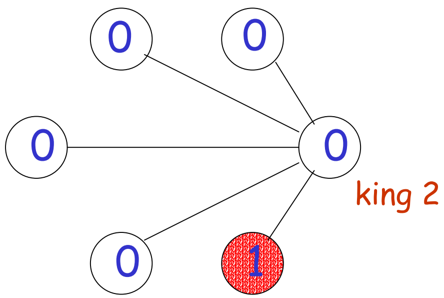
On round 2, everybody will choose the king's value

Phase 2, Round 2



The king broadcasts

Phase 2, Round 2



Everybody chooses the king's value

Final decision

Invariant / Conclusion

In the round where the king is non-faulty, everybody will choose the king's value **v**

After that round, the majority will remain value **v** with a majority population which is at least $n - f > \frac{n}{2} + f$

Exponential Algorithm

solves consensus with n processes and f failures where $f < n/3$ in $f+1$ "phases"

But: uses messages with exponential size

Atomic Broadcast

- One process wants to broadcast message to all other processes
- Either everybody should receive the (same) message, or nobody should receive the message
- Closely related to Consensus: First send the message to all, then agree!

Summary

- We have solved consensus in a variety of models; particularly we have seen
  - algorithms
  - wrong algorithms
  - lower bounds
  - impossibility results
  - reductions
  - etc.

Questions?

Distributed Computing Group
Roger Wattenhofer